Design and Analysis of Algorithms, Chennai Mathematical Institute
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Module – 01
Lecture - 18
Introduction to Graphs

So, when we design an algorithm for a problem, we need to represent the information of the problem in a way that we can manipulate. So, this is called modeling. So, we need some kind of notation and structures to model the property. So, in this module we will look at a very important class of structures called graphs which are immensely useful in many different contexts.

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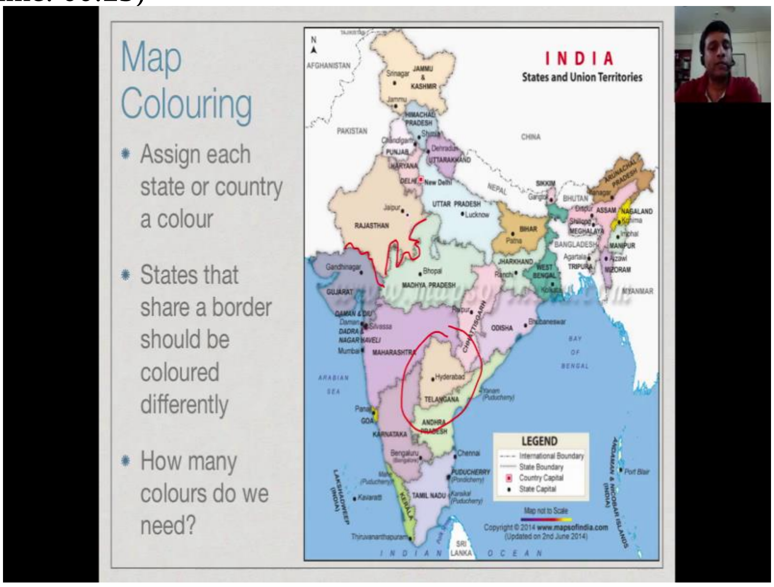

So, let us start with the problem of coloring a political map. So, here is a map of the states of India, as you can see the different states have different colors. Now, what is the principle behind this coloring? When we color a state, we must make sure that no state adjacent to it has the same color, no two state which has side by side that share a common boundary should have the same color, because otherwise it is difficult for us to distinguish one state from another.

So, we can see that Rajasthan has the different color from Gujarat, because Rajasthan and Gujarat at this point share a common boundary. Similarly, Rajasthan and Madhya Pradesh have a boundary, therefore Rajasthan and Madhya Pradesh have a different colors. On the other hand, Rajasthan and Telangana do not share a boundary, so we can use a same color for Rajasthan and Telangana.

So, question we may ask is given such a map, how many colors do we need to color it to satisfy this criteria. Now, clearly we can assign every state a different color and thus we can ensure that no two states which have a boundary in common, in fact no two states anywhere in the map have the same color. But, in many situation we might expect to do with much fewer than that many colors. So, our question is, how many colors do we need?

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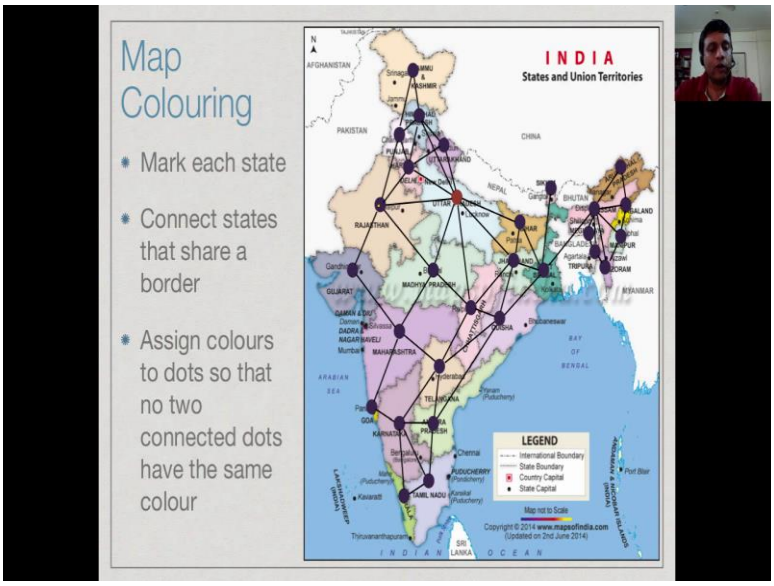

So, here is the way to represent this problem more abstractly. So, the first thing we do is that we replace, we do not replace, we actually assign to each state in the map, a black dot. So, roughly where the state, center of the state we place this black dot. Now, we have to record the information about how the states are neighbors of each other. So, what we do is we connect every pair of states that share a boundary.

So, as we saw Rajasthan and Gujarat have a boundary in common, a border and so we put this kind of an edge between Rajasthan and Gujarat, I am just mark. Now, our coloring problem is to assign colors to the states, so that no two states which share an edge which are on opposite size of an edge have the same color. So, coloring the state is same as coloring the dot associated with the state.

So, we can start for example, with state like Uttar Pradesh and give it a red color and now this is red, it means that state which are near it such as Uttarakhand, or Haryana, or Rajasthan, or Madhya Pradesh, or Chhattisgarh, or Jharkhand, or Bihar. None of these can now be colored red, because they are all neighboring states. So, in the next step having colored Uttar Pradesh red, we have now colored Haryana with different colors say blue and Uttrakhand which has the common boundary with both Haryana and Uttar Pradesh is now colored green, so we have used three colors.

If we proceed with Rajasthan, we find that Rajasthan does not share a boundary with Uttrakhand. So, it is possible to reuse the green color, so we are sticking with three colors, we can extend that coloring to Rajasthan. Then, we get to Punjab, now Punjab is not even connected to Uttar Pradesh, so we can reuse the red color of Uttar Pradesh for Punjab. However, we may come to Himachal Pradesh, we find that Himachal Pradesh is connected to Punjab and Haryana and Uttrakhand and therefore, we have already used up these three colors and we must use a fourth color for Himachal Pradesh.

So, if you keep continuing in this way, we can eventually assign colors to all the states in the country which satisfy this particular property. That whenever, I see two dots, if they are connected by a line, then the two dots are different colors, this captures abstractly the property that you want any two states with a common border, to have different colors on the map.

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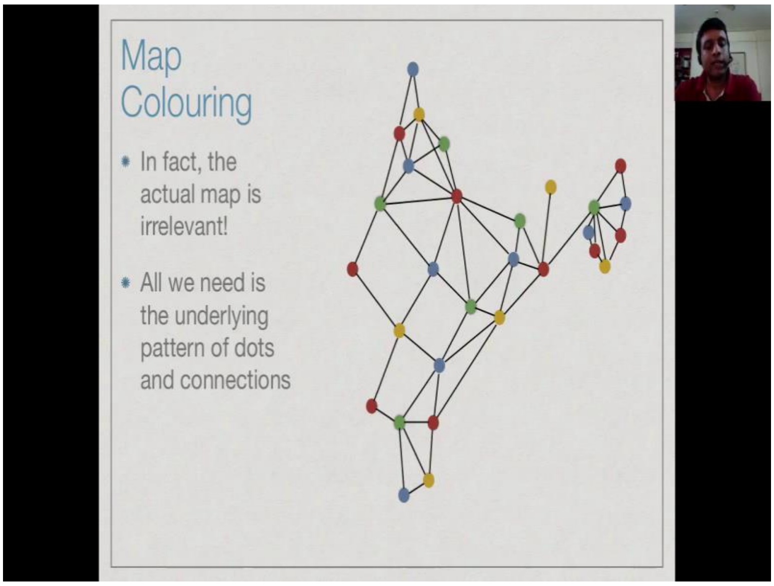

So, now if you look at this map coloring problem, the actual map underline this dot is now not necessary anymore, because the dots captured everything about the map. So, we can actually throw away the underline picture, I am just keep this pattern of dots and the connections between them and if you assign colors to these dots, keeping in mind that these lines are represent common boundaries. We are solving the same problems as solving the original map coloring property.

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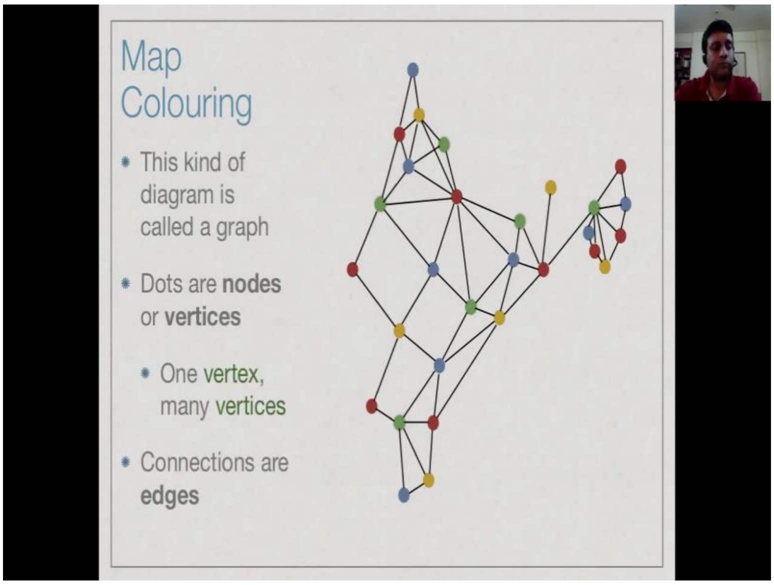

So, this kind of a diagram, this is what we call a graph, so these dots are called vertices, so these are the vertices, they are also called nodes. So, there are two words to describe this, nodes and vertices. It is useful to know that vertices the plural of vertex, the vertex is one node, many nodes are called vertices and these connections are the edges. So, we have the edges between vertices, so it is very simple. A graph is just a picture, it consists of some dots which are nodes of vertices and some connections between them which are called edges.

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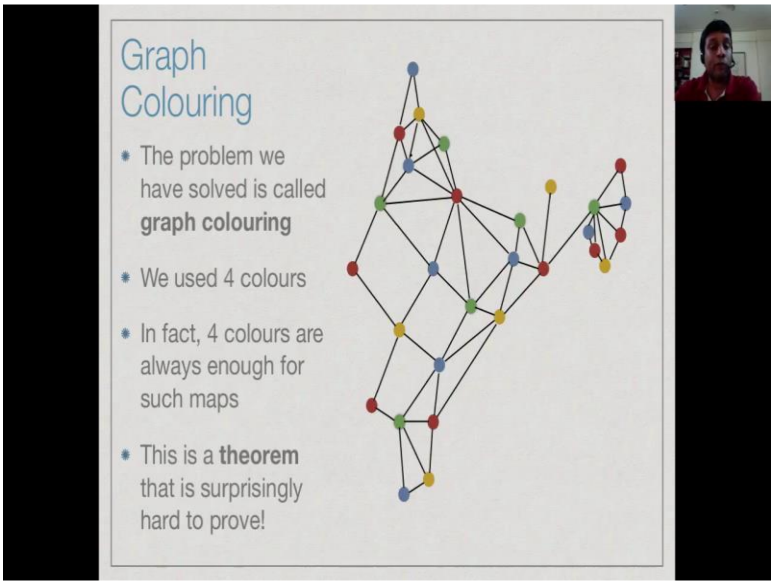

So, the problem that we have solved is called graph coloring. So, we used four colors, you can check that we have used four colors, we have used blue, green, red and yellow and we are managed to color this entire graph with just these four colors. Now, you might ask is it a property of this graph or is it a property of all graph. So, in fact it turns out that if you take any map of the kind that we drew and convert into a graph like we did, four colors are always enough.

Now, this is a mathematical fact about graph. So, you can take a particular problem about coloring a particular map and then we can ask a general question about all maps of this kind and we can actually prove a mathematical fact about it or a theorem, it says that in any map which is derived from this kind of a, any graphs is it derive from this kind of map, four colors are enough to solve the graph coloring property. Now, this is not an easy problem to solve, it was an open problem for many years and it is a very celebrated theorem, when it was actually proved.

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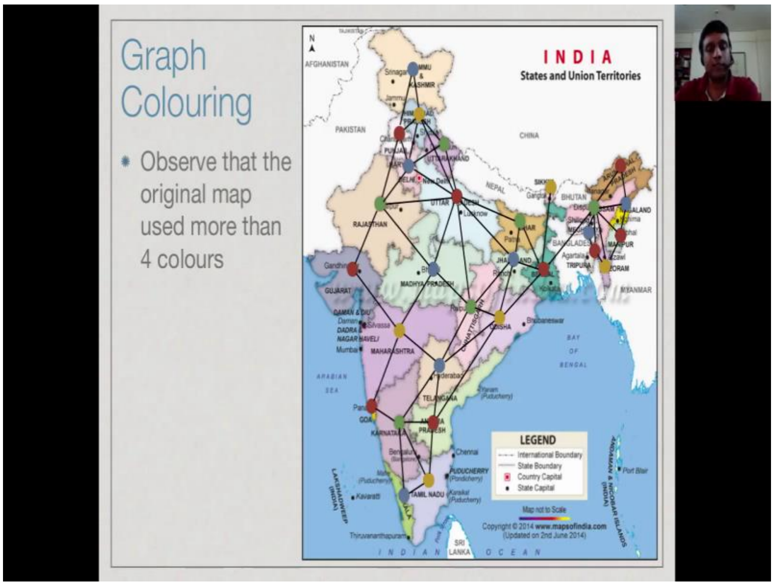

Now, you might observe that though we have found a coloring of this map using our graph with only four colors, the original map which we started with had many more and four colors. So, we have this, if you look around, you will find 1, 2, 3, 4 and then 5 at least five colors on this map, 6 if you include this white and so on. So, though we have at theorem that we can use four colors, in actual practice that person who has colored this map has use many more than four colors.

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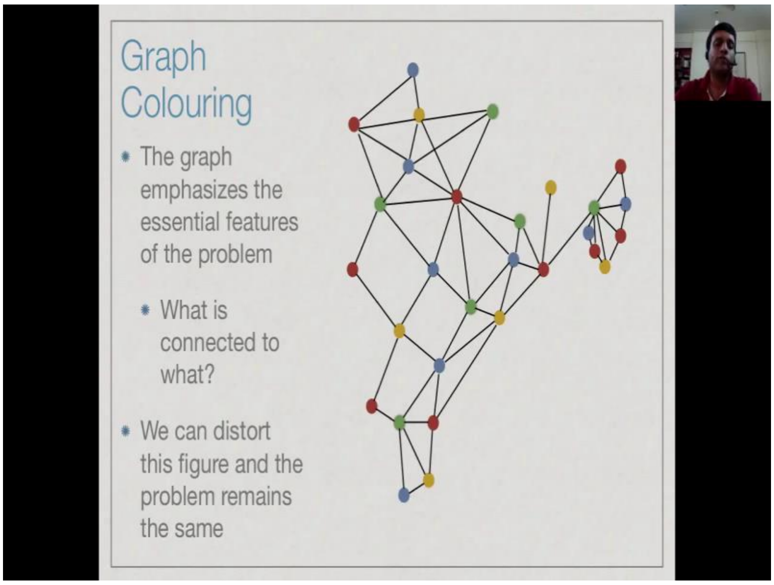

So, one of the advantages of moving to our representation such as a graph is that we have thrown away all the in essential feature of the problem, we do not need to know the shape of this state, we do not need to know it is size. We just need to know which state is connected to which state, that is which state is the border of which state. So, now once we have the graph, then we can redraw the graph.

But, in sense we can expand out these kinds of crowded portions to make the connections more obvious and we can work with this modified graph and the solution to this modified graph is exactly the same as the solution to the original graph, the problem does not change. So, one of the most important features of modeling a problem is to keep the essential parts of the problem and to throw away the inessential part, so that you can focus on the problem that actually needs to be solved.

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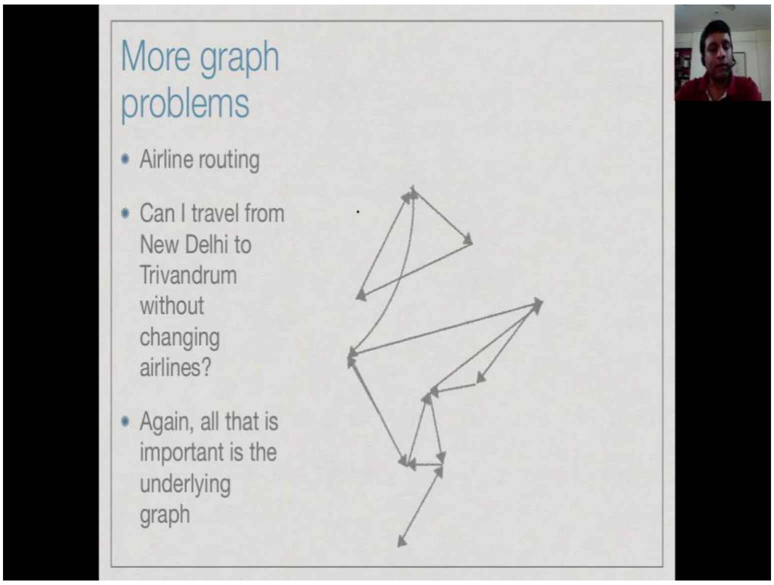

So, another problem that we have seen which is easily representable as a graph is that of an airline routing. So, in we have airline routing, you might ask given the routes of an airline which are represented and this kind of format by cites and arrows between them indicating flights in one direction, you might ask questions about connectivity. Then, I go from New Delhi to Trivandrum without changing airlines.

So, this is again a problem, where we can focus on the connectivity and throw away the actual graph, it does not really a matter to us the actual map, it does not matter to us how far a ((Refer Time: 08:14)) cities are where, they are respect to each other. All we are interested in knowing is which city is connected to which city by a flight and given this what kind of routes can I find between pairs of cities.

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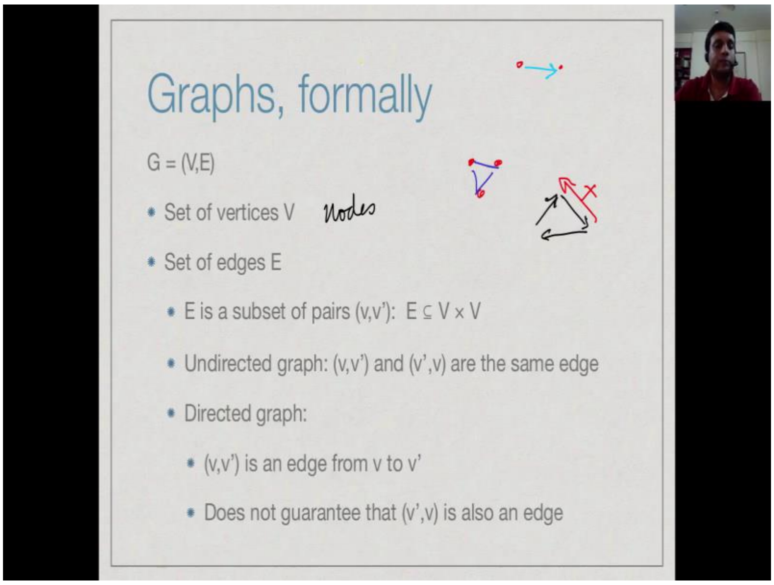

So, formally a graph just consists of two parts, it is a set of vertices or nodes which is normally written V and there are set of edges which are pairs of vertices. So, each edge is the pair v comma v pair. Now, if you have a graph of the kind that we had for the map there we have coloring it, then we do not distinguish whether v is before v prime or v prime is before v. When, we say that two states share a common boundary, it does not matter which order we mention there mean.

So, there is an edge between v and v prime, if and only if V there is an edge between v prime and v, so there is only one edge between any pair of vertices. On the other hand, in the airline graph we had directions, we might have a flight from one city to other city, but not bad. We have this triangular kind of things, where we could go from one city to other there and back.

So, this does not mean that you cannot go back directly on this direction, so this is not directed graph. So, if you have an edge from v to v prime in directed graph, we do not guarantee that there is an edge from v prime to v. And as we saw, we can now easily describe these kinds of graphs as picture by just drawing the nodes and then connecting the edges as lines. So, either undirected it or we could have directed graph in which we draw lines with arrows to indicate the correction.

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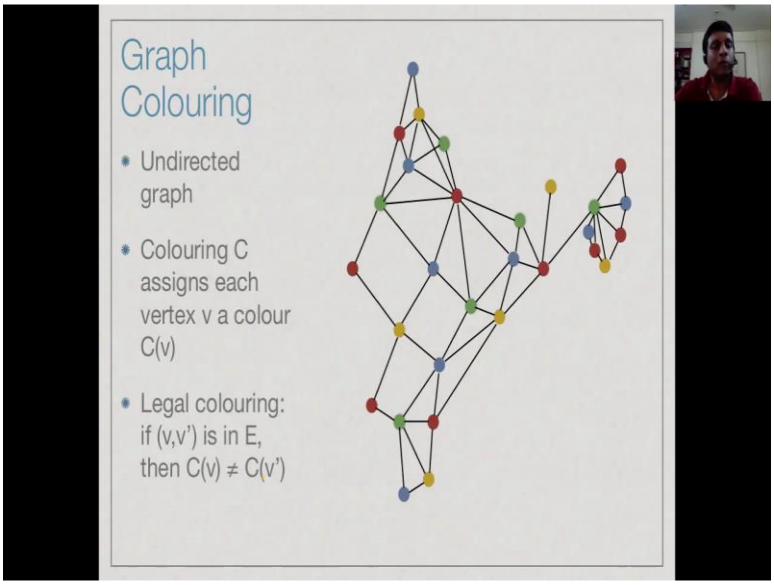

So, the graph coloring problem is a problem on an undirected graph and we can formally say that the problem is to find a coloring C, a coloring C is a function that assigns to each vertex v a color C of v and in terms of the graph, when we have an edge v and v prime in our edge set, then C of v should be different from C of v time. This is what it means to be a legal coloring. So, we can now take this graph as a mathematical object and describe the problem to be solved in a completely objective way as a mathematical problem in terms of the vertices and the edges.

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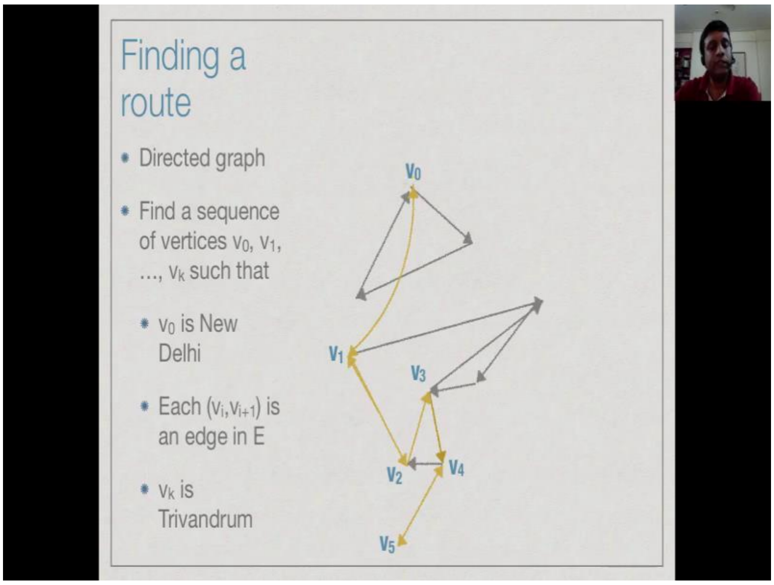

Similarly, we can express the problem of finding a root in the mathematical science, initially our problem is a directed graph. Then, we identify the vertices corresponding to the cities, where we want to find the nodes. So, I give a we have v 0 represent New Delhi and v 5 represent Trivandrum, and our goal was to find the root from v 0 to v 5 from New Delhi to Trivandrum. So, such a root can be described as a path, so path is just a sequence of vertices connected by edges. So, you want to find a sequence of vertices such a first vertex v 0 is where we want to start, the last vertex v k is where we want to end and each pair of vertices along the part v i, v i plus 1 is a flight is an edge in Agra.

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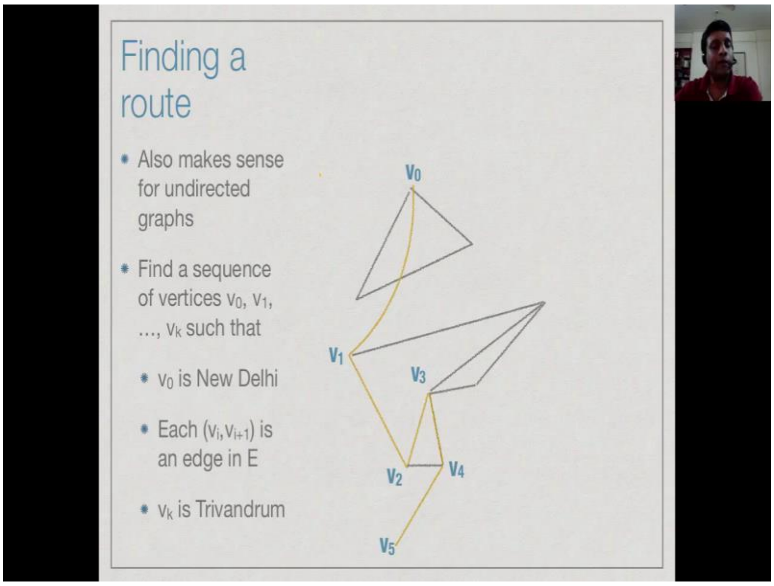

Now, it is not require that a graph needs to be directed for this problem to make sense, we could take the same graph and we could assume they are undirected, that is the airline actually serves all pairs of city in both directions, in the problem still make sense. So, now, we have an undirected graph, but we say that v 0, v 1 is an edge, means v 0, v 1 is also an edge. We can go backwards and forwards and the same question now, can we find the sequence of vertices, we start with v 0 and goes to v 5 such that every pair on this path, every v i, v i, plus 1 is an edge in E