Chapter 10

STATIC INDETERMINANCY; FLEXIBILITY METHOD

`All the examples in this chapter are taken verbatim from White, Gergely and Sexmith`

10.1 Introduction

1 A statically indeterminate structure has more unknowns than equations of equilibrium (and equations of conditions if applicable).

2 The advantages of a statically indeterminate structures are:

1. Lower internal forces
2. Safety in redundancy, i.e. if a support or members fails, the structure can *redistribute* its internal forces to accomodate the changing B.C. without resulting in a sudden failure.

3 Only disadvantage is that it is more complicated to analyse.

4 Analysis mehtods of statically indeterminate structures *must satisfy* three requirements

Equilibrium

Force-displacement (or stress-strain) relations (linear elastic in this course).

Compatibility of displacements (i.e. no discontinuity)

5 This can be achieved through two classes of solution

Force or Flexibility method;

Displacement or Stiffness method

6 The flexibility method is first illustrated by the following problem of a statically indeterminate cable structure in which a rigid plate is supported by two aluminum cables and a steel one. We seek to determine the force in each cable, Fig. 10.1

1. We have three unknowns and only two independent equations of equilibrium. Hence the problem is statically indeterminate to the first degree.

6. We observe that the solution of this sproblem, contrarily to statically determinate ones, depends on the elastic properties.

7 Another example is the propped cantilever beam of length L, Fig. 10.2

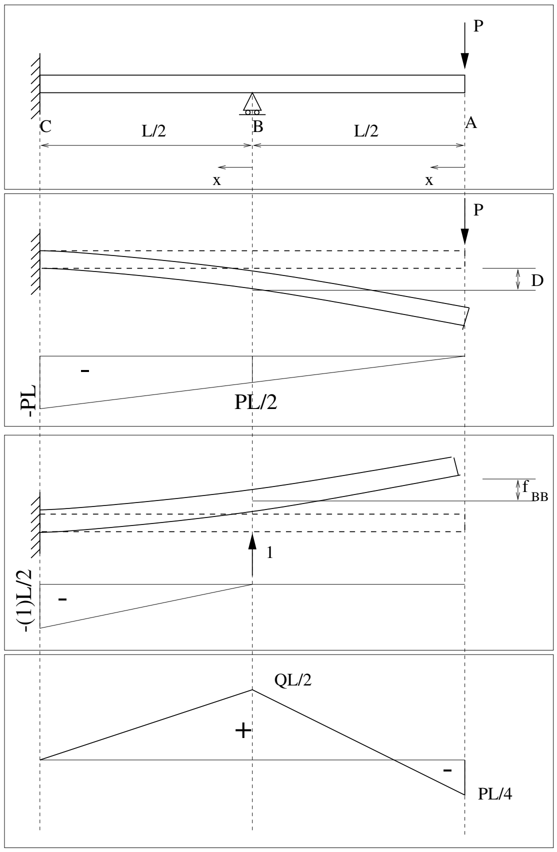


Figure 10.2: Propped Cantilever Beam

1. First we remove the roller support, and are left with the cantilever as a primary structure.
2. We then determine the deflection at point B due to the applied load P using the virtual force method

$$1.D = \int \delta\overline{M}\frac{M}{EI}dx \tag{10.6-a}$$

$$= \int_0^{L/2} 0\frac{-px}{EI}dx + \int_0^{L/2} -\left(\frac{PL}{2}+Px\right)(-x)dx \tag{10.6-b}$$

$$= \frac{1}{EI}\int_0^{L/2}\left(\frac{PL}{2}x+Px^2\right)dx \tag{10.6-c}$$

$$= \frac{1}{EI}\left[\frac{PLx^2}{4}+\frac{Px^3}{3}\right]\Bigg|_0^{L/2} \tag{10.6-d}$$

$$= \frac{5}{48}\frac{PL^3}{EI} \tag{10.6-e}$$

Note that $\mathbf{D}_i^0$ is the vector of initial displacements, which is usually zero unless we have an initial displacement of the support (such as support settlement).

7. The reactions are then obtained by simply inverting the flexibility matrix.

9 Note that from Maxwell-Betti's reciprocal theorem, the flexibility matrix $[\mathbf{f}]$ is always symmetric.

10.3 Short-Cut for Displacement Evaluation

10 Since deflections due to flexural effects must be determined numerous times in the flexibility method, Table 10.1 may simplify some of the evaluation of the internal strain energy. *You are strongly discouraged to use this table while still at school!.*

$g_1(x)$	$g_2(x)$: a, L	$g_2(x)$: a, L	$g_2(x)$: a, b, L
c, L	Lac	$\frac{Lac}{2}$	$\frac{Lc(a+b)}{2}$
c, L	$\frac{Lac}{2}$	$\frac{Lac}{3}$	$\frac{Lc(2a+b)}{6}$
c, L	$\frac{Lac}{2}$	$\frac{Lac}{6}$	$\frac{Lc(a+2b)}{6}$
c, d, L	$\frac{La(c+d)}{2}$	$\frac{La(2c+d)}{6}$	$\frac{La(2c+d)+Lb(c+2d)}{6}$
c, d, e, L	$\frac{La(c+4d+e)}{6}$	$\frac{La(c+2d)}{6}$	$\frac{La(c+2d)+Lb(2d+e)}{6}$

Table 10.1: Table of $\int_0^L g_1(x)g_2(x)dx$

10.4 Examples

■ Example 10-1: Steel Building Frame Analysis, (White et al. 1976)

A small, mass-produced industrial building, Fig. 10.3, is to be framed in structural steel with a typical cross section as shown below. The engineer is considering three different designs for the frame: (a) for poor or unknown soil conditions, the foundations for the frame may not be able to develop any dependable horizontal forces at its bases. In this case the idealized base conditions are a hinge at one of the bases and a roller at the other; (b) for excellent

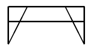
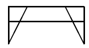
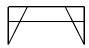
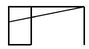
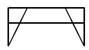
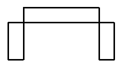
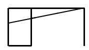
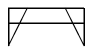
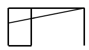
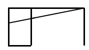
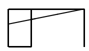
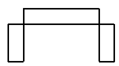
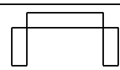
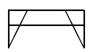
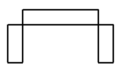
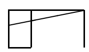
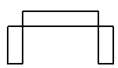
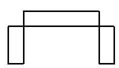
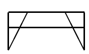
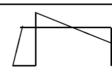
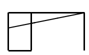
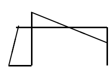
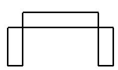
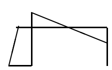
D	$\delta\overline{M}$	M	$\int \delta\overline{M} M dx$			$\int \delta\overline{M} \frac{M}{EI} dx$
			AB	BC	CD	Total
f_{11}			$+\frac{h^3}{3}$	$+Lh^2$	$+\frac{h^3}{3}$	$+\frac{2h^3}{3EI_c} + \frac{Lh^2}{EI_b}$
f_{12}			$+\frac{h^2L}{2}$	$+\frac{L^2h}{2}$	0	$+\frac{h^2L}{2EI_c} + \frac{L^2h}{2EI_b}$
f_{13}			$-\frac{h^2}{2}$	$-Lh$	$-\frac{h^2}{2}$	$-\frac{h^2}{EI_c} - \frac{Lh}{EI_b}$
f_{21}			$+\frac{h^2L}{2}$	$+\frac{L^2h}{2}$	0	$+\frac{h^2L}{2EI_c} + \frac{L^2h}{2EI_b}$
f_{22}			$+L^2h$	$+\frac{L^3}{3}$	0	$+\frac{L^2h}{EI_c} + \frac{L^3}{3EI_b}$
f_{23}			$-hL$	$-\frac{L^2}{2}$	0	$-\frac{hL}{EI_c} - \frac{L^2}{2EI_b}$
f_{31}			$-\frac{h^2}{2}$	$-Lh$	$-\frac{h^2}{2}$	$-\frac{h^2}{EI_c} - \frac{Lh}{EI_b}$
f_{32}			$-hL$	$-\frac{L^2}{2}$	0	$-\frac{hL}{EI_c} - \frac{L^2}{2EI_b}$
f_{33}			$+h$	$+L$	$+h$	$+\frac{2h}{EI_c} + \frac{L}{EI_b}$
D_{1Q}			$-\frac{h^2(2h+15L-30)}{6}$	$+\frac{Lh(20-5L)}{2}$	0	$-\frac{h^2(2h+15L-30)}{6EI_c} + \frac{Lh(20-5L)}{2EI_b}$
D_{2Q}			$-\frac{Lh(2h+10L-20)}{2}$	$+\frac{L^2(30-10L)}{6}$	0	$-\frac{Lh(2h+10L-20)}{2EI_c} + \frac{L^2(30-10L)}{6EI_b}$
D_{3Q}			$+\frac{h(2h+10L-20)}{2}$	$-\frac{L(20-5L)}{2}$	0	$-\frac{h(2h+10L-20)}{2EI_c} - \frac{L(20-5L)}{2EI_b}$

Table 10.3: Displacement Computations for a Rectangular Frame

3. In the following discussion the contributions to displacements due to axial strain are denoted with a single prime ($\prime$) and those due to curvature by a double prime ($\prime\prime$).
4. Consider the axial strain first. A unit length of frame member shortens as a result of the temperature decrease from 85°F to 45°F at the middepth of the member. The strain is therefore

$$\alpha \Delta T = (0.0000055)(40) = 0.00022 \tag{10.56}$$

5. The effect of axial strain on the relative displacements needs little analysis. The horizontal member shortens by an amount $(0.00022)(20) = 0.0044$ ft. The shortening of the vertical members results in no relative displacement in the vertical direction 2. No rotation occurs.
6. We therefore have $D'_{1\Delta} = -0.0044$ ft, $D'_{2\Delta} = 0$, and $D'_{3\Delta} = 0$.

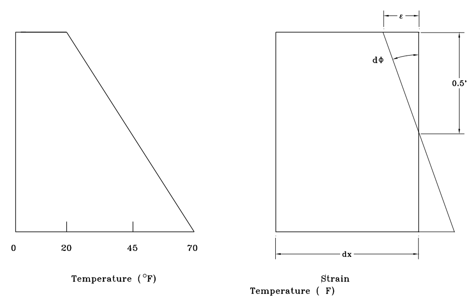


Figure 10.16:

7. The effect of curvature must also be considered. A frame element of length dx undergoes an angular strain as a result of the temperature gradient as indicated in Figure 10.16. The change in length at an extreme fiber is

$$\epsilon = \alpha \Delta T dx = 0.0000055(25)dx = 0.000138dx \tag{10.57}$$

8. with the resulting real rotation of the cross section

$$d\phi = \epsilon/0.5 = 0.000138dx/0.5 = 0.000276dx \text{ radians} \tag{10.58}$$

9. The relative displacements of the primary structure at D are found by the virtual force method.
10. A virtual force $\delta\overline{Q}$ is applied in the direction of the desired displacement and the resulting moment diagram $\delta\overline{M}$ determined.
11. The virtual work equation

$$\delta\overline{Q}D = \int \delta\overline{M} d\phi \tag{10.59}$$

is used to obtain each of the desired displacements D.
12. The results, which you should verify, are

$$D''_{1\Delta} = \boxed{0.0828 \text{ ft}} \tag{10.60-a}$$

$$D''_{2\Delta} = \boxed{0.1104 \text{ ft}} \tag{10.60-b}$$

$$D''_{3\Delta} = \boxed{-0.01104 \text{ radians}} \tag{10.60-c}$$

with units in kips and feet.

21. A moment diagram may now be constructed, and other internal force quantities computed from the now known values of the redundants. The redundants have been valuated separately for effects of temperature and foundation settlement. These effects may be combined with those due to loading using the principle of superposition. ■

■ Example 10-9: Braced Bent with Loads and Temperature Change, (White et al. 1976)

The truss shown in Figure 10.17 reperesents an internal braced bent in an enclosed shed, with lateral loads of 20 kN at the panel points. A temperature drop of 30°C may occur on the outer members (members 1-2, 2-3, 3-4, 4-5, and 5-6). We wish to analyze the truss for the loading and for the temperature effect.

Solution:

1. The first step in the analysis is the definition of the two redundants. The choice of forces in diagonals 2-4 and 1-5 as redundants facilitates the computations because some of the load effects are easy to analyze. Figure **??**-b shows the definition of R_1 and R_2.

2. The computations are organized in tabular form in Table 10.4. The first column gives the bar forces P in the primary structure caused by the actual loads. Forces are in kN. Column 2 gives the force in each bar caused by a unit load (1 kN) corresponding to release 1. These are denoted p_1 and also represent the bar force $\bar{q}_1/\delta\overline{Q}_1$ caused by a virtual force $\delta\overline{Q}_1$ applied

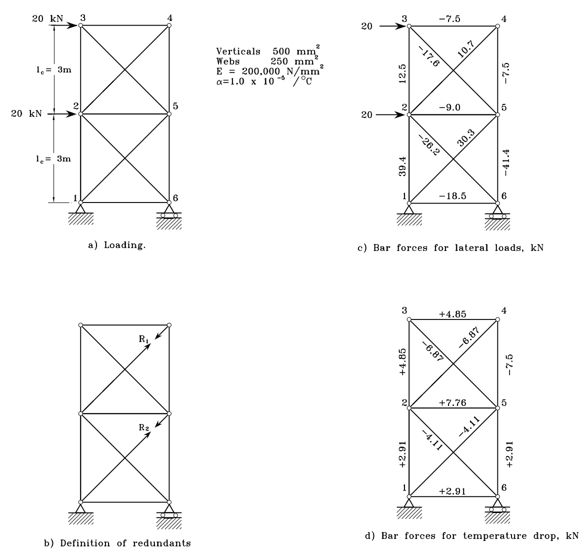


Figure 10.17:

at the same location. Column 3 lists the same quantity for a unit load and for a virtual force $\delta\overline{Q}_2$ applied at release 2. These three columns constitute a record of the truss analysis needed for this problem.

					D_{1Q}	D_{2Q}	f_{11}	f_{21}	f_{12}	f_{22}		$D_{1\Delta}$	$D_{2\Delta}$
	P	p_1	p_2	L/EA	$\bar{q}_1 PL/EA$	$\bar{q}_2 PL/EA$	$\bar{q}_1 p_1 L/EA$	$\bar{q}_2 p_2 L/EA$	$\bar{q} p_2 L/EA$	$\bar{q}_2 p_2 L/EA$	$\Delta/_{temp}$	$\bar{q}_1 \Delta/$	$\bar{q}_2 \Delta/$
multiply by				L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c/EA_c	L_c	$10^{-4}L_c$	$10^{-4}L_c$
1-2	60.0	0	−0.707	1	0	−42.42	0	0	0	0.50	−0.0003	0	2.12
2-3	20.00	−0.707	0	1	−14.14	0	0.50	0	0	0	−0.0003	2.12	0
3-4	0	−0.707	0	2	0	0	1.00	0	0	0	−0.0003	2.12	0
4-5	0	−0.707	0	1	0	0	0.50	0	0	0	−0.0003	2.12	0
5-6	−20.00	0	−0.707	1	0	14.14	0	0	0	0.50	−0.0003	0	2.12
6-1	40.00	0	−0.707	2	0	−56.56	0	0	0	1.00	0	0	0
2-5	20.00	−0.707	−0.707	2	−28.28	−28.28	1.00	1.00	1.00	1.00	0	0	0
1-5	0	0	1.00	2.828	0	0	0	0	0	2.83	0	0	0
2-6	−56.56	0	1.00	2.828	0	−160.00	0	0	0	2.83	0	0	0
2-4	0	1.00	0	2.828	0	0	2.83	0	0	0	0	0	0
3-5	−28.28	1.00	0	2.838	−80.00	0	2.83	0	0	0	0	0	0
					−122.42	−273.12	8.66	1.00	1.00	8.66		6.36	4.24