Chapter 33: Diagonalization

Introduction

Diagonalization is a powerful technique in linear algebra that simplifies matrix operations by converting a given square matrix into a diagonal matrix through similarity transformation. For civil engineers, diagonalization is particularly important in areas such as structural analysis, vibration analysis, and systems of differential equations. A diagonal matrix is easier to work with computationally, especially for raising a matrix to powers or solving systems of equations. Understanding when and how a matrix is diagonalizable allows engineers to interpret and simplify complex real-world models mathematically.

33.1 Diagonalization of a Matrix

A square matrix A of order $n \times n$ is said to be **diagonalizable** if there exists an invertible matrix P and a diagonal matrix D such that:

$$A = P D P^{-1}$$

Here,

- A is the original matrix,
- D is the diagonal matrix,
- P is the matrix whose columns are the **linearly independent eigenvectors** of A,
- D contains the **eigenvalues** of A along its diagonal.

This is called a **similarity transformation**, and it allows easier computation for powers of A:

$$A^k = P D^k P^{-1}$$

33.2 Eigenvalues and Eigenvectors Review

To diagonalize a matrix, you must first find its **eigenvalues** and **eigenvectors**.

Given an $n \times n$ matrix A, a non-zero vector $\vec{v}$ is called an **eigenvector** if:

$$A\vec{v} = \lambda\vec{v}$$

where λ is a scalar called the **eigenvalue** corresponding to $\vec{v}$.

To find eigenvalues:

$$\text{Solve } det(A - \lambda I) = 0$$

This is called the **characteristic equation**.

To find eigenvectors:

- For each eigenvalue λ, solve the equation:

$$(A - \lambda I)\vec{v} = 0$$

33.3 Diagonalization Criteria

A matrix A is **diagonalizable** if and only if:

- It has n **linearly independent** eigenvectors.
- Equivalently, the algebraic multiplicity (how many times an eigenvalue appears as a root of the characteristic polynomial) must equal the geometric multiplicity (dimension of the eigenspace) for each eigenvalue.

Special Cases:

- **Distinct Eigenvalues**: If all eigenvalues are distinct, then A is always diagonalizable.
- **Repeated Eigenvalues**: Matrix may or may not be diagonalizable; check the number of linearly independent eigenvectors.

33.4 Procedure to Diagonalize a Matrix

Given a square matrix A, follow these steps:

Step 1: Find the characteristic polynomial:

$$det(A - \lambda I) = 0$$

Step 2: Find all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Step 3: For each eigenvalue λ_i, find the null space of $(A - \lambda_i I)$ to get the corresponding eigenvectors.

Step 4: Form matrix P using linearly independent eigenvectors as columns.

Step 5: Construct diagonal matrix D with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ along the diagonal.

Step 6: Check invertibility of P and compute P^{-1}.

Then,

$$A = P D P^{-1}$$

33.5 Example

Let's diagonalize the matrix:

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

Step 1: Characteristic polynomial:

$$det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & 1 \\ 2 & 3 - \lambda \end{vmatrix} = (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10$$

Solve:

$$\lambda^2 - 7\lambda + 10 = 0 \Rightarrow \lambda = 5, 2$$

Step 2: Eigenvectors.

For $\lambda = 5$:

$$(A - 5I)\vec{v} = 0 \Rightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 2$:

$$(A - 2I)\vec{v} = 0 \Rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Step 3: Construct P and D:

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$$

Check:

$$A = P D P^{-1}$$

Hence, matrix A is diagonalizable.

33.6 Applications in Civil Engineering

Diagonalization plays a key role in:

- **Vibration Analysis**: Natural frequencies of structures (e.g., bridges or buildings) are found using eigenvalues of system matrices.
- **Structural Dynamics**: Solving second-order differential equations in multiple degrees of freedom systems.
- **Stability of Structures**: Eigenvalue methods help in buckling analysis.
- **Finite Element Methods (FEM)**: Diagonalizing stiffness or mass matrices for numerical solutions.
- **Markov Processes in Transport Engineering**: Transition matrices in modeling traffic flow or urban planning.

33.7 Non-Diagonalizable Matrices and Jordan Form (Brief Note)

Some matrices cannot be diagonalized because they don't have enough linearly independent eigenvectors. Such matrices can still be analyzed using **Jordan canonical form**, which is a generalization of diagonalization. However, for most physical systems modeled in civil engineering, matrices are usually symmetric or well-behaved, and thus **diagonalizable**.

33.8 Diagonalization of Symmetric Matrices

A matrix A is **symmetric** if $A^T = A$. In the case of **real symmetric matrices**, the diagonalization process has some special properties:

- All eigenvalues of a symmetric matrix are **real**.
- Eigenvectors corresponding to **distinct eigenvalues** are **orthogonal**.
- A real symmetric matrix is always **orthogonally diagonalizable**: That is,

  $$A = Q D Q^T$$

  where:

- Q is an orthogonal matrix ($Q^T = Q^{-1}$),
- D is a diagonal matrix with real eigenvalues.

Why this is useful:

In structural engineering, stiffness matrices are symmetric due to physical principles. Orthogonal diagonalization ensures numerical stability and simplified modal analysis.

33.9 Numerical Aspects in Diagonalization

In practical computations (especially using MATLAB, Python, or structural engineering software), diagonalization is used under the hood to solve systems efficiently.

Computational Cautions:

- Not all matrices are diagonalizable—if a matrix has fewer than n linearly independent eigenvectors, it cannot be diagonalized.
- If eigenvalues are very close together, numerical methods may introduce rounding errors—important in high-precision structural modeling.

Software Tools for Eigen Computation:

- **MATLAB:**

```
[P,D] = eig(A)
```

- **Python (NumPy):**

```
import numpy as np
eigenvalues, P = np.linalg.eig(A)
D = np.diag(eigenvalues)
```

- **Scilab/Octave/Excel** also provide eigenvalue functions.

33.10 Repeated Eigenvalues and Geometric Multiplicity

Let's dig deeper into **multiplicity**:

- **Algebraic multiplicity (AM)**: Number of times an eigenvalue appears in the characteristic equation.

- **Geometric multiplicity (GM)**: Number of linearly independent eigenvectors associated with that eigenvalue.

Key Rule: A matrix A is diagonalizable if **GM = AM** for all eigenvalues.

Example of Non-Diagonalizable Matrix

$$A=\begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}$$

- Characteristic polynomial: ¿
- Eigenvalue: $\lambda=2$, AM = 2
- $A-2I=\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$

Solving $(A-2I)v=0$ gives **only one** linearly independent eigenvector ⇒ GM = 1.

So, A is **not diagonalizable**.

33.11 Diagonalization and Matrix Powers

One of the major applications of diagonalization is computing matrix powers efficiently:

Let:

$$A=PDP^{-1}\Rightarrow A^k=PD^kP^{-1}$$

Where D^k is just the diagonal matrix with each diagonal entry raised to the power k :

$$D=\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\0 & \lambda_2 & \cdots & 0\\\vdots & \vdots & \ddots & \vdots\\0 & 0 & \cdots & \lambda_n\end{bmatrix}\Rightarrow D^k=\begin{bmatrix}\lambda_1^k & 0 & \cdots & 0\\0 & \lambda_2^k & \cdots & 0\\\vdots & \vdots & \ddots & \vdots\\0 & 0 & \cdots & \lambda_n^k\end{bmatrix}$$

This is computationally much faster and accurate in simulations involving repeated transformations, dynamic systems, and state-transition models.

33.12 Physical Interpretation in Structural Systems

In civil engineering, especially in **multi-degree-of-freedom (MDOF)** systems like frames and trusses:

- The **mass matrix** and **stiffness matrix** are symmetric and positive definite.
- Diagonalizing the stiffness matrix helps decouple the system into **independent single-degree-of-freedom systems**.
- Each eigenvalue corresponds to a **natural frequency**.
- Each eigenvector represents a **mode shape** (how the structure vibrates at that frequency).

This process is the foundation of **modal analysis**, a critical part of earthquake engineering and dynamic response analysis.

33.13 Practice Problems

1. Diagonalize the following matrix if possible:

$$A = \begin{bmatrix} 6 & -2 \\ 2 & 2 \end{bmatrix}$$

2. Determine whether the matrix is diagonalizable:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

3. Show that every real symmetric 2×2 matrix is diagonalizable with real eigenvalues.

4. A structure modeled with the stiffness matrix

$$K = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$$

undergoes modal analysis. Find its eigenvalues and interpret the result in terms of mode shapes.
