Solid Mechanics

Prof. Ajeet Kumar

Deptt. of Applied Mechanics

IIT, Delhi

Lecture - 9

Mohr's Circle

Welcome to Lecture 9! In this lecture, we will discuss about the very interesting concept of Mohr's circle.

1 Conditions for applying Mohr's Circle (start time: 00:26)

Mohr's circle is a graphical way to find normal and shear components of traction on arbitrary planes. The only restriction is that the plane normal has to be perpendicular to one of the principal stress directions. Accordingly, let us consider a coordinate system such that the third coordinate axis is along one of the principal stress directions (the rest two coordinate axes need not be along any principal direction, also see Figure 1) and we are looking at planes whose normals are perpendicular to the third principal direction. Then, all such plane normals will have $n_3 = 0$. On such planes, we want to find the normal and shear components of traction (see Figure 2). The stress matrix in this coordinate system will be such that its column will be formed by traction on the plane along third coordinate axis. But that being a principal plane, the third column will not have any shear component. Hence, the third column will have its last entry nonzero but the other two zero. Symmetry of stress matrix will further force the first two entries of last row to be zero, i.e.,

$$[\underline{\sigma}] = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{bmatrix} \tag{1}$$

Also, the normal vectors of the plane on which we want to obtain normal and shear components will look like

$$[\underline{n}] = \begin{bmatrix} \times \\ \times \\ 0 \end{bmatrix} \tag{2}$$

2 Deriving formulas for normal and shear components on such planes (start time: 04:19)

Let us draw a cuboid element at the point of interest with its face normals along the coordinate system (see Figure 1). As discussed earlier, the third coordinate axis ($\underline{e}_3$) is along the third principal direction. We will call $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$ axes as x, y and z axes, respectively. On the $\underline{e}_1$ plane, we have normal component of traction denoted as σ_{xx} and shear component of traction denoted as τ_{yx} pointing towards y axis. The third component (τ_{zx}) is absent as per the stress representation (1). Similarly, on the $\underline{e}_2$ plane, we have τ_{xy} (same as τ_{yx}) and on the $\underline{e}_3$ plane, we have σ_{zz} (equal to λ_3) only. On the $-\underline{e}_1$ plane, the normal component is σ_{xx} in $-x$ direction and shear component is τ_{yx} in $-y$ direction and likewise for other two negative planes.

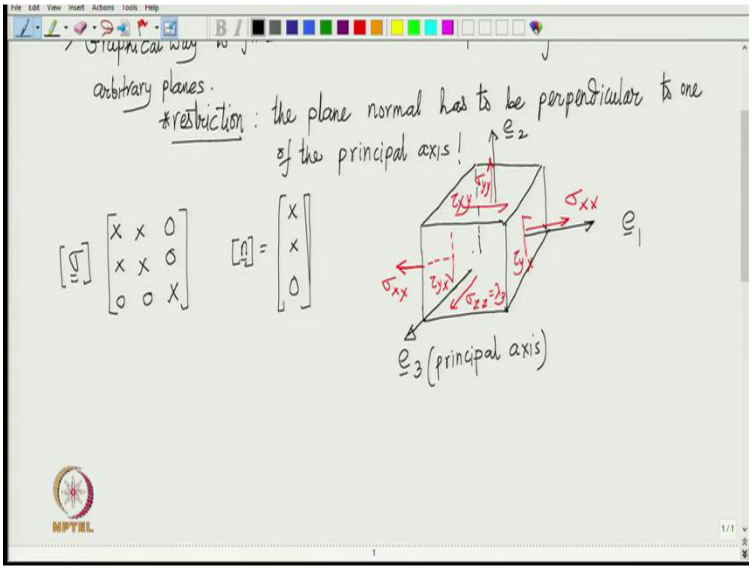

Figure 1: The cuboid with face normals along the three coordinate axes. The third coordinate axis is also a principal direction. The traction components are also shown.

2.1 Reducing the cuboidal representation of state of stress to a square (start time: 07:53)

There is another simpler way to draw such a state of stress for whom there are no shear components in the third direction: we can just draw a square instead of a cube where the sides of the square denote the faces of the cuboid as shown in Figure 2. The right edge of the square represents the (+$\underline{e}_1$) face. Similarly, the top edge represents +$\underline{e}_2$ face and the plane of the square itself represents $\underline{e}_3$ face. So, the plane containing the square has just one traction component σ_{zz}. We use a dot enclosed by a circle to denote the traction component coming out of the plane as shown in Figure 2. On the edges of the square, we have both the shear and the normal components of traction. On the right edge, representing $\underline{e}_1$ face, we have σ_{xx} and τ_{yx} (which is equal to τ_{xy} in magnitude). So, we will just write τ_{xy} to denote both τ_{yx} and τ_{xy}. On the top edge, representing $\underline{e}_2$ face, we have τ_{xy} and σ_{yy}. Keep in mind that such a reduce to square to represent stress matrix is possible only when the third coordinate axis lies along one of the principal directions.

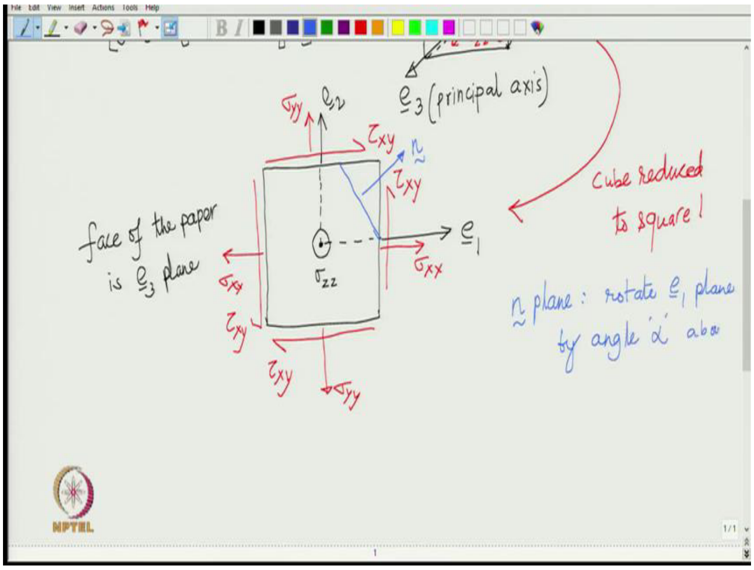

Figure 2: A square with its sides representing the faces of the cuboid. Traction components are also drawn.

2.2 Trigonometric formula for σ and τ (start time: 11:04)

Now, our goal is to calculate the normal and shear components of traction on planes whose normals are perpendicular to $\underline{e}_3$. The blue line in Figure 2 shows a general plane of such kind. The normal to this plane is represented by $\underline{n}$ and assume that it makes an angle α with $\underline{e}_1$ axis. To get to this arbitrary plane $\underline{n}$, we can rotate our $\underline{e}_1$ plane by α about the $\underline{e}_3$ axis. The column representation of $\underline{n}$ in this coordinate system ($\underline{e}_1, \underline{e}_2, \underline{e}_3$) will be

$$[\underline{n}] = \begin{bmatrix} cos(\alpha) \\ sin(\alpha) \\ 0 \end{bmatrix} \tag{3}$$

An important point to note here is that we have assumed the direction of τ on this plane makes 90° (anti clockwise) from the normal vector $\underline{n}$ (also see Figure 2). Now, σ will be given by

$$\begin{aligned} \sigma = \left([\underline{\sigma}]\ [\underline{n}] \cdot [\underline{n}] \right) &= \left(\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \begin{bmatrix} cos(\alpha) \\ sin(\alpha) \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} cos(\alpha) \\ sin(\alpha) \\ 0 \end{bmatrix} \\ &= \sigma_{xx} cos^2(\alpha) + 2\tau_{xy} sin(\alpha) cos(\alpha) + \sigma_{yy} sin^2(\alpha) \end{aligned} \tag{4}$$

To get τ, we need to first represent the direction along which it is acting which we denote by $\underline{n}^{\perp}$. If we look at Figure 2, we can find the angle that $\underline{n}^{\perp}$ makes with all the three axes. The representation of $\underline{n}^{\perp}$ will thus be

$$\underline{n}^{\perp} = \begin{bmatrix} -sin(\alpha) \\ cos(\alpha) \\ 0 \end{bmatrix} \tag{5}$$

Now, to get τ, we need to take the component of total traction ($[\underline{\sigma}]$ $[\underline{n}]$) on this plane along the $\underline{n}^{\perp}$ direction, i.e.,

$$\tau = \left([\underline{\sigma}]\ [\underline{n}] \right) \cdot [\underline{n}^{\perp}] \quad = \quad \left(\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \begin{bmatrix} cos(\alpha) \\ sin(\alpha) \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} -sin(\alpha) \\ cos(\alpha) \\ 0 \end{bmatrix}$$

$$= \quad -\sigma_{xx} cos(\alpha) sin(\alpha) - \tau_{xy} sin^2(\alpha) + \tau_{xy} cos^2(\alpha) + \sigma_{yy} cos(\alpha) sin(\alpha) \qquad (6)$$

Upon doing some algebraic manipulation in equation (4), we get

$$\sigma = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sigma_{xx}\left(cos^2(\alpha) - \frac{1}{2}\right) + \sigma_{yy}\left(sin^2(\alpha) - \frac{1}{2}\right) + 2\tau_{xy} sin(\alpha) cos(\alpha) \qquad (7)$$

Further, using the following trigonometric identities:

$$cos(2\alpha) \quad = \quad cos^2(\alpha) - sin^2(\alpha) = 2cos^2(\alpha) - 1 = 1 - 2sin^2(\alpha), \qquad (8)$$

$$sin(2\alpha) \quad = \quad 2sin(\alpha) cos(\alpha), \qquad (9)$$

we obtain

$$\begin{aligned} \sigma &= \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sigma_{xx}\frac{cos(2\alpha)}{2} - \sigma_{yy}\frac{cos(2\alpha)}{2} + \tau_{xy} sin(2\alpha) \\ &= \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} cos(2\alpha) + \tau_{xy} sin(2\alpha) \end{aligned} \qquad (10)$$

Similar rearrangements and simplifications using trigonometric identities (8) and (9) in equation (6) gives us

$$\tau = -\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right) sin(2\alpha) + \tau_{xy} cos(2\alpha) \qquad (11)$$

2.3 Introducing Graphical Parameters (start time: 23:42)

If we look at equations (10) and (11), we notice that σ and τ both have $cos(2\alpha)$ and $sin(2\alpha)$ terms. Let us define a scalar R as

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \qquad (12)$$

We can now think of a right angled triangle with hypotenuse R and the two perpendicular arms as $\frac{\sigma_{xx} - \sigma_{yy}}{2}$ and τ_{xy} as shown in Figure 3. Let us denote the angle between the hypotenuse and the base as 2φ. Thus, from basic trigonometry, we see that

$$cos(2\phi) = \frac{\sigma_{xx} - \sigma_{yy}}{2R}, \quad sin(2\phi) = \frac{\tau_{xy}}{R}, \quad tan(2\phi) = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \qquad (13)$$

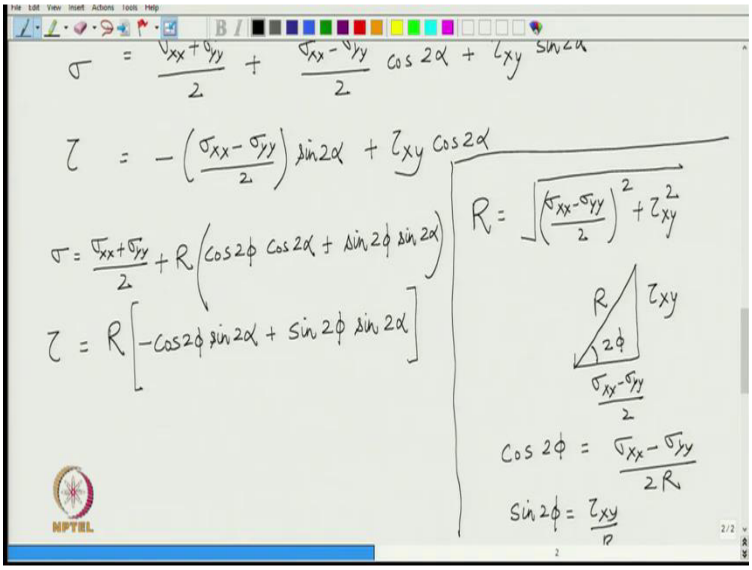

Figure 3: A right angled triangle with the two arms as $\dfrac{\sigma_{xx} - \sigma_{yy}}{2}$ and τ_{xy}.

Using equation (13) in equations (10) and (11), we get

$$\begin{aligned}\sigma &= \frac{\sigma_{xx} + \sigma_{yy}}{2} + R\Big(cos(2\phi)cos(2\alpha) + sin(2\phi)sin(2\alpha)\Big)\\ \tau &= R\Big[-cos(2\phi)sin(2\alpha) + sin(2\phi)sin(2\alpha)\Big]\end{aligned} \quad (14)$$

Upon further using following trigonometric identities:

$$cos(a - b) = cos(a)cos(b) + sin(a)sin(b), \quad (15)$$

$$sin(a - b) = sin(a)cos(b) - cos(a)sin(b), \quad (16)$$

we get

$$\boxed{\begin{aligned}\sigma &= \frac{\sigma_{xx} + \sigma_{yy}}{2} + R\,cos(2\varphi - 2\alpha)\\ \tau &= R\,sin(2\varphi - 2\alpha)\end{aligned}} \quad (17)$$

These are the formulae obtained for getting σ and τ on a plane making an angle α with $\underline{e}_1$ axis. For a given stress matrix, we can find out R and φ using equations (12) and (13) respectively. Then, using equation (17), we can find σ and τ on the plane which is obtained by rotating $\underline{e}_1$ by angle α about $\underline{e}_3$ axis. Let us try to represent the above two formulas graphically.

3 Graphical representation of the derived formulation (start time: 31:07)

We need to see what equation (17) means. Let us think of a $\sigma - \tau$ plane and plot σ and τ for each α in this plane. The plane with σ on the x-axis and τ on the y-axis is shown in Figure 4. From equation (17), we can see that by plotting all points, we will get a circle centered on the σ axis at $\left(\dfrac{\sigma_{xx} + \sigma_{yy}}{2}, 0\right)$. Let us start by plotting σ_{xx} and σ_{yy} on the σ axis. The center of the circle is thus at the mid point of the two points. We also plot τ_{xy} on the τ axis. Now, we can plot the point (σ_{xx}, τ_{xy}) which corresponds to $\underline{e}_1$ plane. If we join this point with the center, the line obtained will give us the radius of the circle. This is because the circle has to pass through the point corresponding to $\underline{e}_1$ plane: the circle is the locus of all (σ, τ) when α is varied and the point corresponding to $\underline{e}_1$ plane is obtained for $\alpha = 0$. We can also verify that the radius R that we obtained graphically also matches with the formula for R in equation (12).

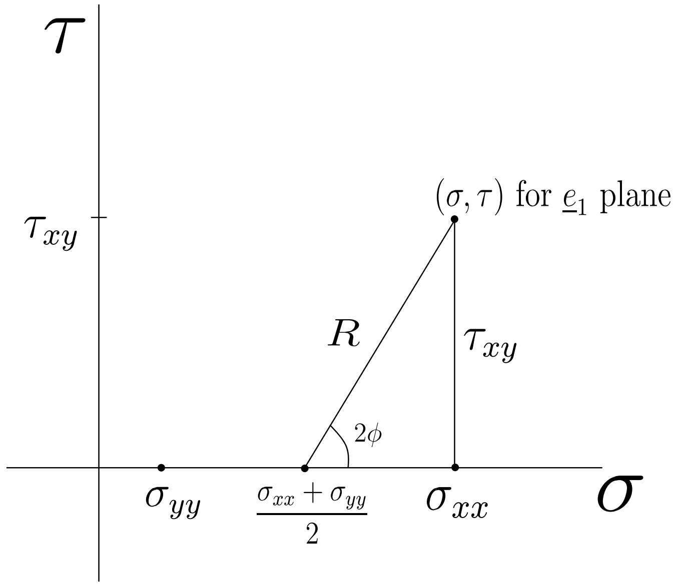

Figure 4: $\sigma - \tau$ plane with various parameters plotted on it.

3.1 Mohr's Circle (start time: 34:09)

Once we have obtained the radius and the center of the circle, we can draw the complete circle as shown in Figure 5. This circle is called the Mohr's circle. When we compare the right angled triangle in Figure 5 with Figure 3, we see that the angle that the line from center to the point corresponding to $\underline{e}_1$ plane makes with the σ axis would be 2φ. To find σ and τ on any arbitrary plane (for general α), let us look at equation (17): the angle in the cosine and sine terms there is ($2\varphi - 2\alpha$). So, for a general plane, the argument in the trigonometric function there reduces by 2α. So, the radial line from the center to the point corresponding to the α-plane on Mohr's circle should be at an angle of ($2\varphi - 2\alpha$) from the x-axis. Thus, we can obtain the point corresponding to α-plane on Mohr's circle by going in the clockwise direction by angle 2α from the $\underline{e}_1$-plane point as shown in Figure 5.

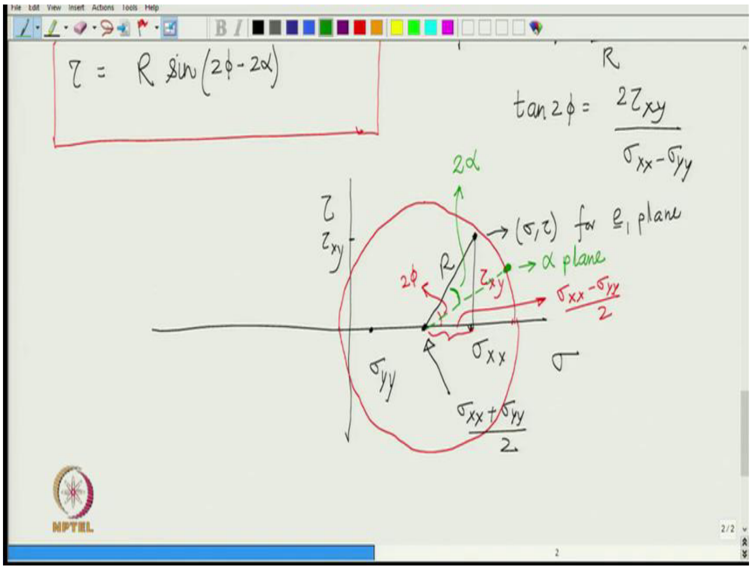

Figure 5: Mohr's circle plot

We summarize below the steps involved in drawing the Mohr's circle and finding the point coresponding to α-plane:

1. Draw the center of the circle at $\left(\frac{\sigma_{xx}+\sigma_{yy}}{2},0\right)$.
2. Draw (σ,τ) for $\underline{e}_1$ plane, i.e. the point (σ_{xx},τ_{xy}).
3. the line joining the center and the point for $\underline{e}_1$ plane forms the radius of the circle.
4. With center and radius known, draw your circle!
5. To find (σ,τ) for α-plane, rotate the radial line of $\underline{e}_1$ plane by 2α clockwise.

Notice that the normal to the required plane made an angle α with $\underline{e}_1$ in the counter clockwise direction (also see Figure 2). But on the Mohr's circle, we draw that point by rotating by 2α in the clockwise direction from the point corresponding to the $\underline{e}_1$ plane. This is because in equation (17), we have 2α with a minus sign in the trigonometric functions. So, clockwise rotation of the plane corresponds to counter-clockwise rotation in the Mohr's circle and vice versa.

4 Sign convention while using Mohr's cirle (start time: 42:40)

Let us draw the Mohr's circle again as shown in Figure 6. We know the point corresponding to the $\underline{e}_1$ plane. To get to the $\underline{e}_2$ plane, α should be 90° in the counter-clockwise direction. So, in the Mohr's circle, we need to go 180° in the clockwise direction from $\underline{e}_1$ plane. Thus, we get the $\underline{e}_2$ plane at the diametrically opposite point with respect to $\underline{e}_1$ plane point. The two right angled triangles in Figure 6 are similar and thus, we get σ at this point as σ_{yy} as required but τ as $-\tau_{xy}$. However, in Figure 2, we see that on $\underline{e}_2$ plane, shear component is τ_{xy}. So, why are we getting $-\tau_{xy}$ from the Mohr's circle? This is because of our convention for the shear component of traction. In Figure 2, we had defined positive τ when we go 90° in the counter clockwise direction from $\underline{n}$. So, to get shear component on $\underline{e}_2$ plane, we go 90° in the anti-clockwise direction from $\underline{e}_2$ direction and thus, get to the $-\underline{e}_1$ direction. So, Mohr's circle is giving us τ on

$\underline{e}_2$ plane in the $-\underline{e}_1$ direction whereas τ_{xy}, by definition, is the shear component in the $+\underline{e}_1$ direction. Therefore, Mohr's circle gives us $-\tau_{xy}$ as the shear traction on $\underline{e}_2$ plane. We can remember that this negative sign comes because $\underline{n}^{\perp}$ for $\underline{e}_2$ plane is along $-\underline{e}_1$ direction by convention.

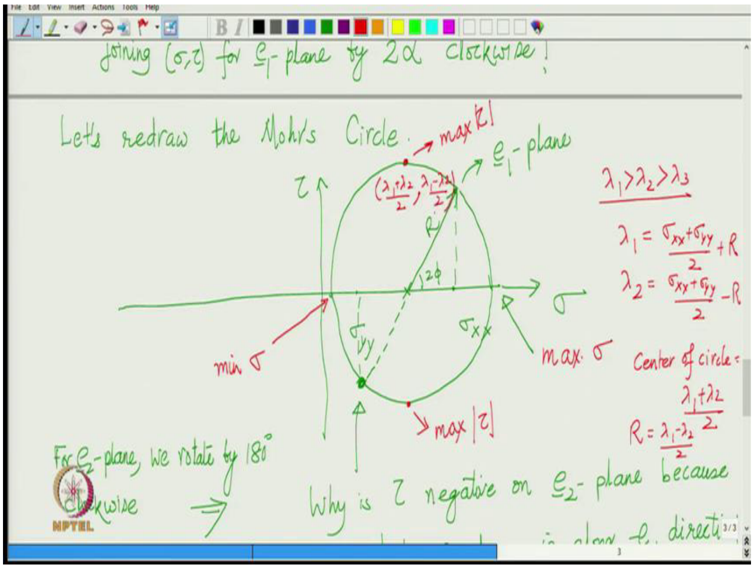

Figure 6: Mohr's circle with important quantities marked on it.

5 Other conclusions that can be drawn using Mohr's circle (start time: 48:57)

We can also get the maximum and minimum values of σ and τ using Mohr's circle. The maximum and minimum values of σ are attained on the σ axis itself. These are plotted in Figure 6. The maximum and minimum values of τ are on the top and bottom of the circle respectively as highlighted in Figure 6. We can verify that these values match with the ones derived in the last lecture. Suppose that the principal stress components λ_1, λ_2 and λ_3 are defined such that $\lambda_1 > \lambda_2 > \lambda_3$. Thus, λ_1 and λ_2 will correspond to the points of maximum σ and minimum σ respectively. From the circle, λ_1 and λ_2 will be obtained by adding and subtracting R to the σ for center respectively, i.e.,

$$\lambda_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + R \tag{18}$$

$$\lambda_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} - R \tag{19}$$

This allows us to get the value of principal stress components directly from the Mohr's circle: otherwise one has to solve an eigenvalue problem. Writing the center of the circle in terms of principal stress components, we obtain

$$\text{Center} = \left(\frac{\lambda_1 + \lambda_2}{2}, 0\right) \tag{20}$$

We can also get the radius of the circle in terms of principal stress components by subtracting equation (19) from equation (18), i.e.,

$$R = \frac{\lambda_1 - \lambda_2}{2} \tag{21}$$

Notice that in the previous lecture, we had derived the maximum value of shear component of traction to be $\frac{\lambda_1 - \lambda_2}{2}$. And from Mohr's circle too, we get $\tau_{\max} = R = \frac{\lambda_1 - \lambda_2}{2}$. From the Mohr's circle, we can also see that σ corresponding to the point where we have maximum shear will be the same as σ for the center of the circle, i.e. $\frac{\lambda_1 + \lambda_2}{2}$. This is the same value that we had derived in the previous lecture. We have thus verified our results of the Mohr's circle.