Chapter 4: Case of Complex Roots

Introduction

In civil engineering, many systems and structures exhibit dynamic behavior—such as buildings swaying during earthquakes or bridges vibrating due to traffic. These behaviors are often modeled using **second-order linear differential equations**. When solving these, the nature of the **characteristic equation's roots** determines the form of the general solution. One particularly interesting and important case arises when the roots are **complex conjugates**. This chapter explores this scenario in depth, explaining the mathematical theory and its civil engineering applications.

4.1 General Form of Second-Order Linear Differential Equations

A **homogeneous second-order linear differential equation with constant coefficients** is of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Where:

- $a, b, c \in \mathbb{R}$ and $a \neq 0$
- $y = y(x)$ is the unknown function

We solve such equations using the **characteristic (auxiliary) equation**:

$$ar^2 + br + c = 0$$

This is a quadratic equation. Its nature depends on the discriminant $D = b^2 - 4ac$.

4.2 Case of Complex Roots (When $D < 0$)

When $D = b^2 - 4ac < 0$, the quadratic equation has **complex conjugate roots**:

$$r = \alpha \pm i\beta$$

Where:

- $\alpha = -\frac{b}{2a}$ (real part)
- $\beta = \frac{\sqrt{4ac - b^2}}{2a}$ (imaginary part)

Thus, the general solution to the differential equation becomes:

$$y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

Where C_1 and C_2 are arbitrary constants determined by initial or boundary conditions.

4.3 Derivation of the Solution

Let us derive this result step-by-step:

Given roots: $r = \alpha \pm i\beta$

So,

$$y(x) = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$$

Using Euler's formula:

$$e^{i\beta x} = \cos \beta x + i \sin \beta x$$

This gives:

$$y(x) = e^{\alpha x} \left[C_1(\cos \beta x + i \sin \beta x) + C_2(\cos \beta x - i \sin \beta x) \right]$$

Grouping terms and setting new constants $A, B \in \mathbb{R}$, we rewrite:

$$y(x) = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$$

This is the required **real-valued solution**.

4.4 Interpretation of the Solution

The solution $y(x) = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$ represents a **damped oscillation**:

- The **exponential factor** $e^{\alpha x}$ (where $\alpha < 0$) causes the amplitude to decay over time.
- The **sine and cosine terms** describe an oscillatory behavior with frequency β.

In Civil Engineering:

- α relates to the damping effect in materials (energy loss due to internal friction or resistance).
- β relates to the **natural frequency** of vibration of structures.

This has real-world importance when analyzing how structures respond to periodic forces (like wind or earthquakes).

4.5 Example Problems

Example 1: Solve:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$$

Solution:

Characteristic equation:

$$r^2 + 4r + 13 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i$$

So the general solution is:

$$y(x) = e^{-2x}(C_1 \cos 3x + C_2 \sin 3x)$$

Example 2 (Application in Civil Engineering): A building undergoes a damped vibration modeled by:

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0$$

With:

- Mass $m = 1$
- Damping coefficient $c = 2$
- Stiffness $k = 5$

Then:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0 \Rightarrow r = -1 \pm 2i \Rightarrow y(t) = e^{-t}(C_1 \cos 2t + C_2 \sin 2t)$$

This tells us the building vibrates at a frequency of 2 rad/s with exponential decay due to damping.

4.6 Engineering Insight: Stability of Structures

In structural dynamics:

- **Stable System**: If $\alpha < 0$, vibrations die out — structure is safe.
- **Unstable System**: If $\alpha > 0$, oscillations grow — potential collapse.

Designing damping systems (like shock absorbers, tuned mass dampers) is essential to control this behavior.

4.7 Real-World Applications in Civil Engineering

1. Earthquake Engineering During an earthquake, buildings experience sudden ground motion, which causes vibrations in the structure. These are typically modeled by second-order differential equations. Complex roots indicate **oscillatory motion**, and their damping rate determines how long the shaking will persist.

> **Example**: A 10-storey RC frame building experiences damped vibrations after an earthquake shock. Engineers analyze the system with damping to ensure vibrations subside within a safety window. The nature of the roots of the characteristic equation helps predict whether the building will remain stable.

2. Design of Suspended Bridges Suspension bridges like the Golden Gate Bridge must withstand wind-induced oscillations. These oscillations can become destructive (resonance) if damping is not properly designed.

- The governing differential equation often yields complex roots.
- Engineers design **aerodynamic dampers** based on the values of α and β.

3. Tall Buildings and Wind Loads Tall skyscrapers sway due to wind pressure. If the sway follows a second-order differential model and yields complex roots, it indicates **periodic motion with decreasing amplitude** (if $\alpha < 0$).

- Structures like Burj Khalifa use **tuned mass dampers** to shift natural frequencies and control β, reducing perceived motion.

4.8 Graphical Representation and Physical Meaning

1. Phase Plot of Damped Oscillations For a solution:

$$y(t) = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$$

The displacement $y(t)$ vs time t plot shows a sinusoidal wave with decreasing amplitude over time — this is known as **exponentially decaying oscillation**.

2. Envelope Curve The exponential term $e^{\alpha t}$ creates an **envelope** over the oscillating waveform, indicating how the amplitude diminishes with time.

Graphical components:

- **Outer envelope**: $\pm e^{\alpha t}$
- **Inner oscillation**: sinusoidal component with frequency β

  Graphical illustration should be included here in your e-book using a plotted graph or simulation showing an oscillating wave with a damping envelope.

4.9 Damping Ratio and Natural Frequency

Damping Ratio ζ

$$\zeta = \frac{c}{2\sqrt{mk}}$$

- If $\zeta < 1$, the system is **underdamped** and exhibits oscillatory motion (complex roots).
- The general solution:

$$y(t) = e^{-\zeta \omega_n t}(C_1 \cos \omega_d t + C_2 \sin \omega_d t)$$

Where:

- $\omega_n = \sqrt{\frac{k}{m}}$ = Natural frequency
- $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ = Damped frequency

Civil Engineering Example: For a damped floor slab system:

- $m = 500$ kg
- $c = 100$ Ns/m
- $k = 20000$ N/m

We calculate:

- $\omega_n = 6.32$ rad/s
- $\zeta = 0.354$ (underdamped)
- $\omega_d = 5.9$ rad/s

This gives insight into how quickly and smoothly the floor returns to equilibrium after a disturbance.

4.10 Numerical Methods: Simulating Complex Root Behavior

When analytical solutions are difficult, civil engineers use **numerical integration methods** like:

- **Runge-Kutta (RK4)**
- **Euler's Method**
- **Finite Difference Method**

Example: Using RK4 in MATLAB or Python to simulate:

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 25y = 0$$

- Converts into a system of first-order ODEs
- Simulates real-time vibration damping in beams or building elements
- Useful in real-time health monitoring systems of structures

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4.11 Experimental Validation and Structural Monitoring

Modern civil engineering incorporates **sensor-based structural health monitoring (SHM)**.

- Accelerometers collect real-time displacement data.
- The measured vibration data is analyzed and modeled using second-order differential equations.
- The nature of the system's response (complex roots or not) helps identify if a structure is damaged or behaving abnormally.

Example:

> A sensor on a bridge deck records a vibration signal that shows a sinusoidal waveform decaying over time. This confirms a complex root behavior — suggesting the structure is still performing within elastic limits.

4.12 Summary of the Chapter

Concept	Explanation
Differential Equation	$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$
Discriminant D	If $D < 0$, roots are complex conjugates
General Solution	$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$
Application	Damped vibrations in structures
Engineering Insight	Helps predict stability and response of civil systems

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Exercises

Q1. Solve the equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 0$

Q2. A structure vibrates with equation $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0$. Plot the solution and identify the damping characteristics.

Q3. A spring-mass-damper system has $m = 2$ kg, $c = 4$ Ns/m, $k = 50$ N/m. Calculate $\zeta, \omega_n, \omega_d$ and classify the response.

Q4. Explain the significance of damping ratio ζ in the design of a tall structure.

4.7 Key Takeaways

- Complex roots occur when $b^2 - 4ac < 0$ in the characteristic equation.
- The solution is a product of an exponential decay and a sinusoidal function.
- This models **damped oscillatory motion**, fundamental in structural analysis.
- Understanding this case helps civil engineers **predict and mitigate structural responses** to dynamic loads.

Conclusion

The case of complex roots in second-order differential equations is central to analyzing and controlling **dynamical systems in civil engineering**. Whether it's ensuring a bridge can withstand traffic-induced vibrations or designing earthquake-resistant buildings, mastering this topic equips engineers with essential tools for structural safety and innovation.