Hydraulic Engineering
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Lecture-08
Basics of fluids Mechanics-II (Contd.,)

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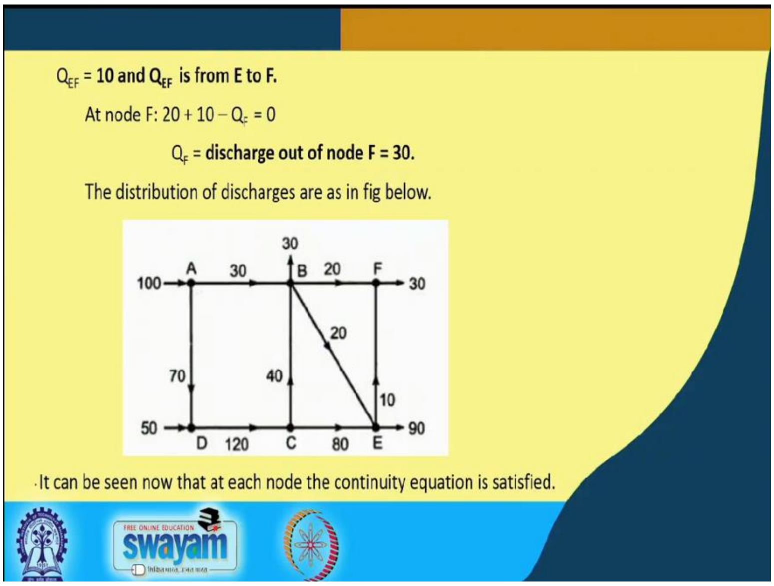

Welcome back to the next lecture. So, last lecture we finished with the equation of continuity and solving a very simple problem for a pipe flow where the discharges were given. So, we have seen it the equation of continuity A 1 V 1 = A 2 V 2.

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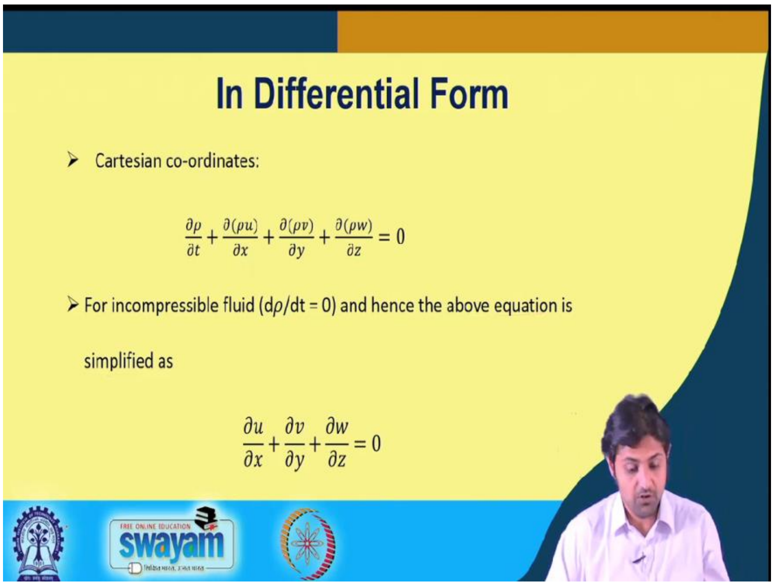

So, now we need to also see it in a differential form, this is the most famous form of the continuity equation. So, in Cartesian coordinates the equation of continuity is written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

these are the convective terms. For incompressible flow this the density does not changes with time and hence the above equation is simplified as and the ρ can come out it does not change either with respect to x or I mean, the coordinate and time. Therefore, this can simply be written as, so, this will go to 0, ρ will come out, so, it will become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

. So, this is the continuity equation in the differential form.

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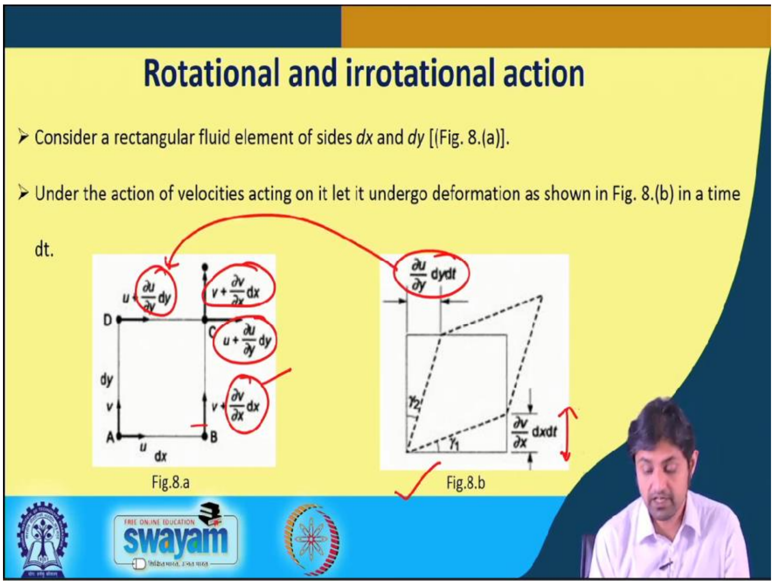

Now, talk about rotational and irrotational action. If we consider, a rectangular fluid element of side dx and dy, as in this figure you see, this is point A, this is point B, here, point A, point B, and this is dx, this is the distance, I mean, this is the distance of, you know, the side, dx and this is the dy. This is a fluid particle. Under the action of velocities, for example, here there is a velocity A, at A it is u and at in the in the x direction and V in the y direction.

So, on the side dx, it is u and this dy its v. So, this, you know, fluid element can undergo a rotation and a rotation or a deformation right now, I mean, we call it deformation as we can see here. So, this will undergo some rotation, you know and so, at point B the velocities if we assume, in this direction it is changing. So, u is changing in this direction as del u / del y into dy and similarly, while coming from here this velocity is changing in x direction as del v del x dx.

Therefore, the u and v corresponding, u and v velocities here, will be u + del u del y dy and v velocity will be $v+\frac{\partial v}{\partial x}dx$. So, this fluid particle will rotate as it appears in the figure 8.b and this distance, this distance is going to be $\frac{\partial v}{\partial x}dxdt$, if we are considering the rotation and time dt. Similarly, this will be coming from here multiplied by dt and if this angle is gamma 1 and this is gamma 2.

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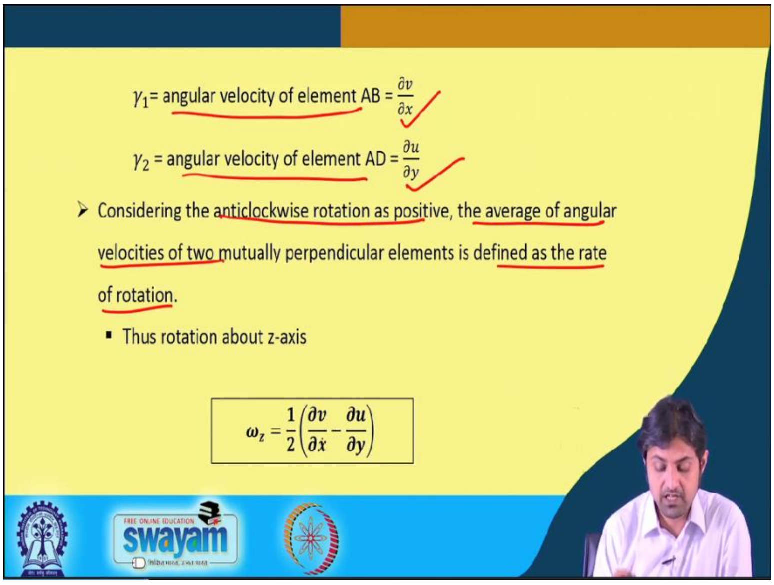

Gamma1 here, is angular velocity of element AB which is equal to del v del x. Gamma 2 is angular velocity of element ad, that is, del u del y. We can go back to this slide here, you see, del v del x and this is del u del y as we have written. Considering the anti clockwise rotation as positive, the average of the angular velocities of the 2 mutually perpendicular elements is defined as rate of rotation.

So, this example, we took to tell you about what actually is the rate of rotation. Thus, the rotation about z axis can be given as half into del v x del v / del x - del u / del y, because this is going to rotate, you see, the rotation this direction and because of this it is in this direction and because of this it is in this direction. So, they are opposite to each other. That is why this negative sign so, it is rotation in z axis is $\boldsymbol{\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}$

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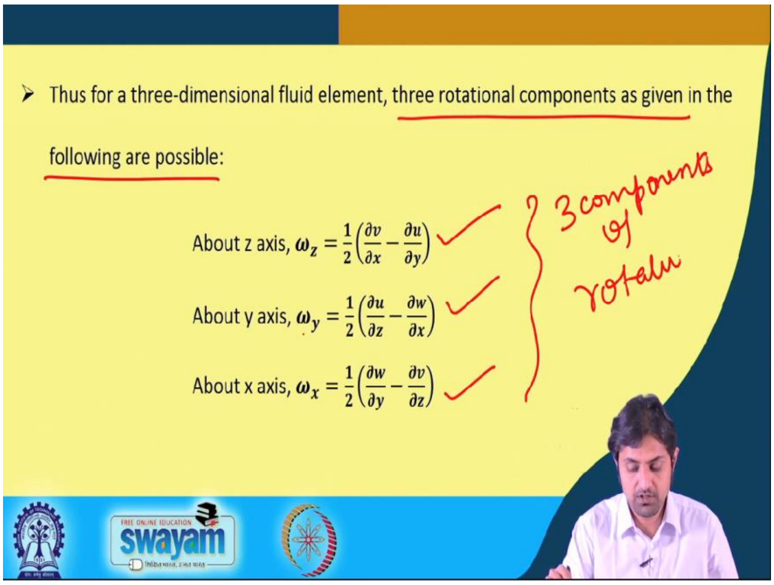

Thus, for a 3 dimensional fluid element 3 rotational components, as given following are possible, about z axis the rotation is given as $\boldsymbol{\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}$. Similarly, about y axis, its rotation is given as $\boldsymbol{\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}$ and repeating the same exercise in X rotation about x axis, it is given as $\boldsymbol{\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}$. So, these are the 3 components of rotation.

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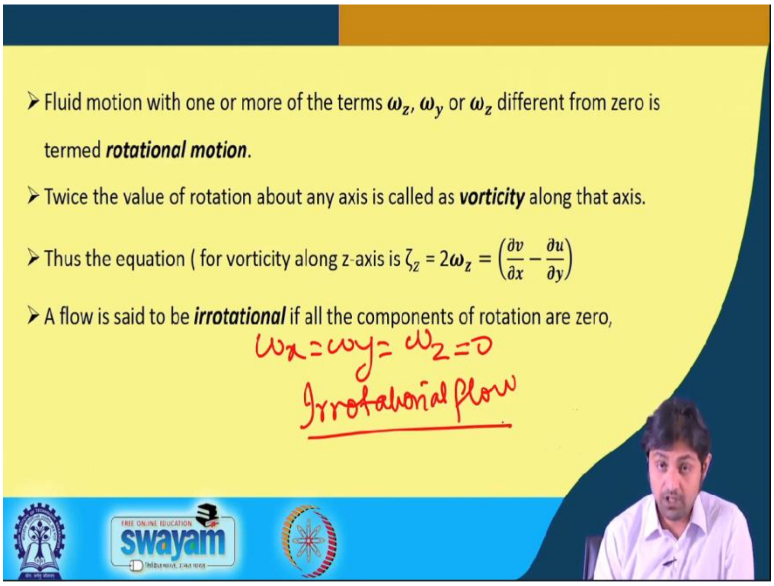

Fluid motion, with one or more of the terms omega z, omega y or omega x. So, this is not omega z, this is omega x. If these terms are not 0, this motion where these omegas at least one of them is not 0 is called rotational motion. So, the condition is omega z slash omega y slash omega x should not be equal to 0, if at least one of them is not equal to 0 it is called rotational motion. So, twice the value of these omegas about any axis is called the vorticity along that axis.

You must have heard the term vorticity. Thus, the equation for vorticity along z axis is 2 omega z is equal to. So, this is if you remember, the omega z was half times this thing. So, this is the vorticity along. So, a flow now is said to be irrotational if all components of rotational is 0. So, that means, omega x is equal to omega y is equal to omega z is equal to 0. This is the definition of irrotational flow and this is what is, let, me write it more properly. Omega x is equal to omega y is equal to omega z is equal to 0.

This is called irrotational flow. This is what we assume, the water flow and in most of our hydraulic engineering based application we assume, the water is irrotational. The flow in the water is irrotational that is why this concept is quite important. You already know this because you have already done the fluid mechanics course. So, exactly what I have written.

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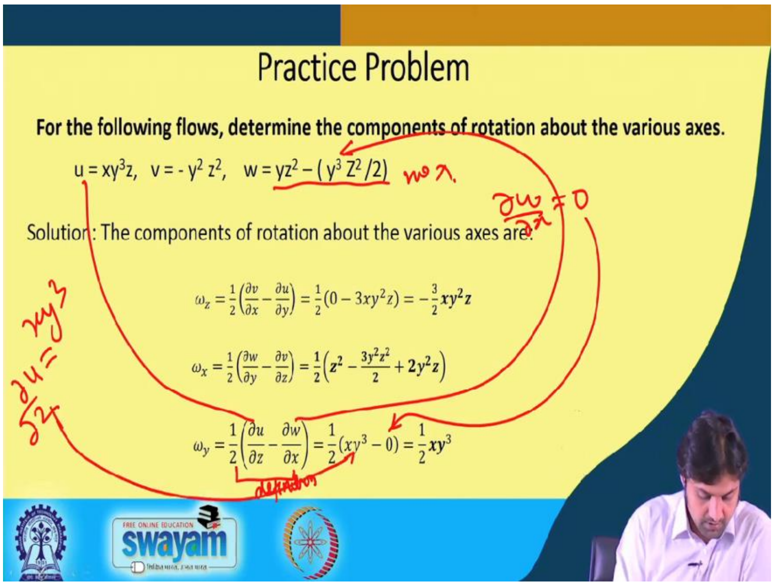

Now, we will see, a practice problem, for the following flows, determine the components of rotation about various axes. So, we have been given u is equal to x y cube z, V is equal to - y square z square, w is yz square - y cube z square / 2. So, we already know the formulas. So, the components of rotation about various axes our omega z, you see here, is given by this is a standard formula that we have seen.

So, what we need to do is del v del x. So, del v is not a function of x. So, del v del x will be 0 here, as it is written, or let me just, you know, take an eraser it is, so, this one. So, this is going to be so, I will bring back the pen, so, this will be 0 now del u / del y. So, from here, del u / del y is going to be 3 x y square z and this is what has been written here, and this 0 has been written here. So, therefore, which implies omega z is written as - 3 / 2x y square z. This is one of the component. So, I am erasing all the ink again.

So, we can concentrate on the other, omega x is by definition. So, again this is definition, we need to calculate del w del y and del v del z. So, let is look at w here , there is a y component. So, del w del y. So, del w del y can be written as z square - 3y square z square / 2 and this is here. See here, now, we also need to know what del v del z is. So, del v / del z, from here, is going to be minus 2 y square z, and this has been put with a minus sign here, so, this becomes plus when coming here.

So, I will erase ink, so, that you are able to follow. So, this is the. And similarly, omega y will be half, this again is a definition. Now, del u del z, so, it will be x y cube and del w del x. So, there is no x here. So, del w del x will be is equal to 0 and this is has been shown here, this has been shown here. So, I will erase all these ink here. So, we have calculated wz, wx and wy and this gives us our 3 rotational components.

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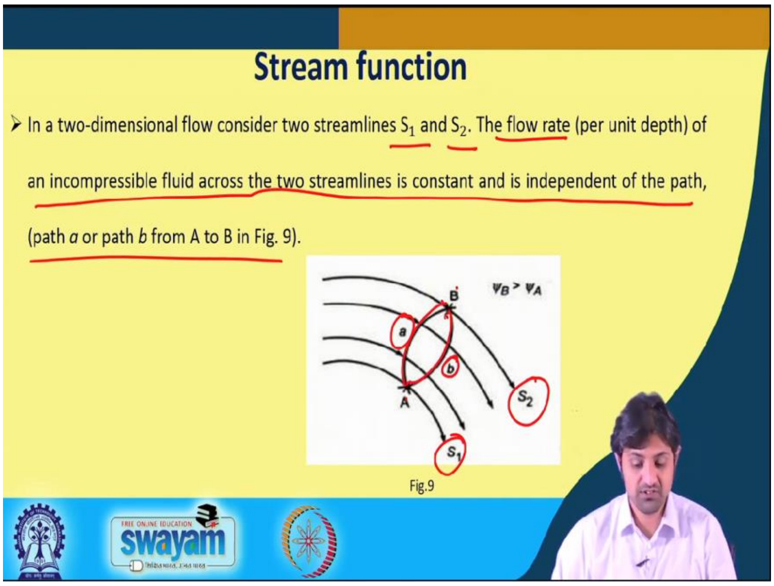

Now, stream function, what actually is a stream function? So, in a 2d flow or a 2 dimensional flow, this type of flow considers two stream lines S1 and S2. The flow rate per unit depth of an incompressible fluid across two stream lines is constant and independent of the path. So, in a 2 dimensional flow we consider two stream lines S1and S2. The flow rate of an incompressible fluid, for example, water across the 2 stream line is constant and is independent of the path. Path a or path b from A to B in figure 9.

So, this is the figure, this is one stream line, this is two stream line. So, you see, this is A, this is B and this is path a, this one going to B and this is path b going away from this to this .

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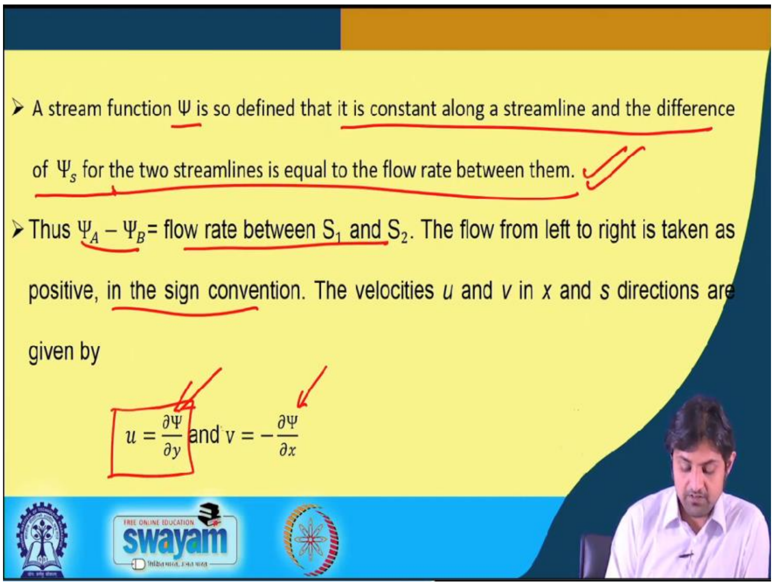

So, based on that a stream function psi is so defined that it is constant along the stream line and the difference of these stream line for the 2 streamline is equal to the flow rate between them. This is an important definition for we will have a lecture on viscous fluid flow where we will be deriving everything from scratch the equation of continuity to Navier Stokes equation along with this stream lines, we are going to see. There we will prove this, but for now, you have as studied in your fluid mechanics class, the stream function phi is defined so that this stream function is constant along the stream line and the difference of the phi s that the stream function for the 2 stream lines A and B is equal to the flow rate between them. Thus, phi A - phi B is equal to flow rate between S 1 and S 2. The flow from left to right is taken as positive, in the sign convention. So, this is what we assumed, that if the flow is from left to right it is positive. The velocity is u and v in x and s directions are given by, this is important. So, if there is a stream function psi. So, u is given by del psi / del y and v is given as del psi / del x. This is the most important formula that actually you must be remembering.

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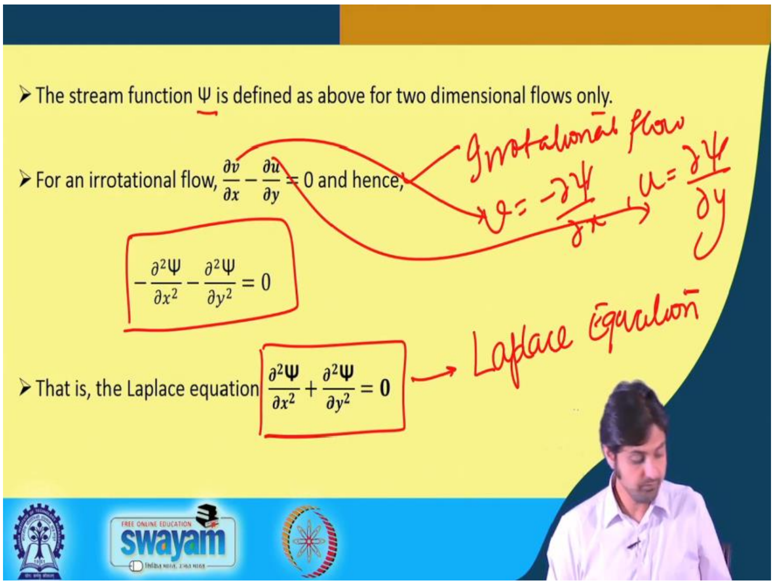

So, the stream function psi is defined as above for 2 dimensional flows only. It cannot be defined for the 3 dimensional, for an irrotational flow del v / del x - del u / del y is equal to 0. This is we know from irrotational flow, correct. Therefore, if we substitute v as, what we saw v was you go back and see v was - del psi / del x. And u was del psi / del y.

Now, put v as this and u as this you are going to get $\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0$ and this is the Laplace equation.

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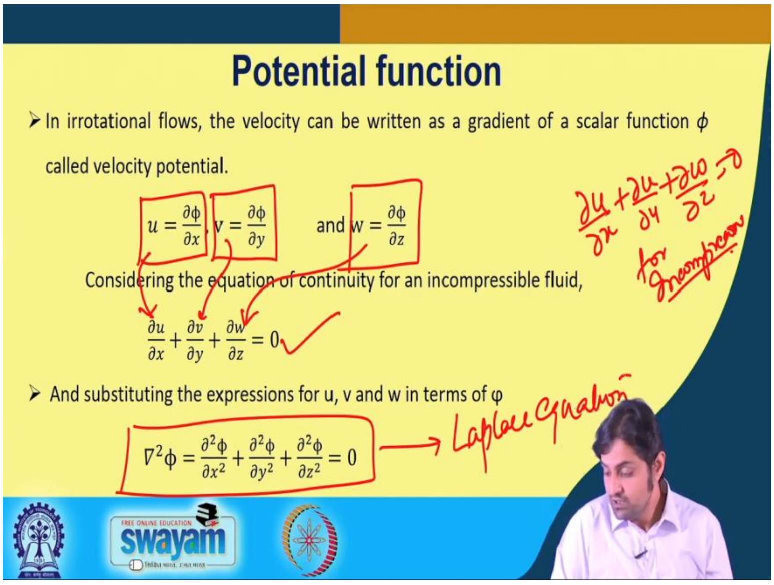

So, we can proceed to the next slide, there is something called a potential function. So, in an irrotational flow the velocity can be written as, gradient of scalar function phi called the velocity potential that is for an irrotational flow. Phi is velocity potential therefore, u in terms of velocity potential is written as del phi / del x, v is written as del phi / del y and w is written as del phi / del z.

Considering, the equation of continuity for an incompressible fluid, what was the equation of continuity? If you recall del u / del x + del v / del y + del w / del z is equal to 0, for incompressible, correct. This we have already written, alright and if we substitute these u v and w here, what are we going to get? del square phi is equal to del square phi / del x square + delta square phi / del y square + del y / del z square and this is again tell me Laplace equation, that is good.

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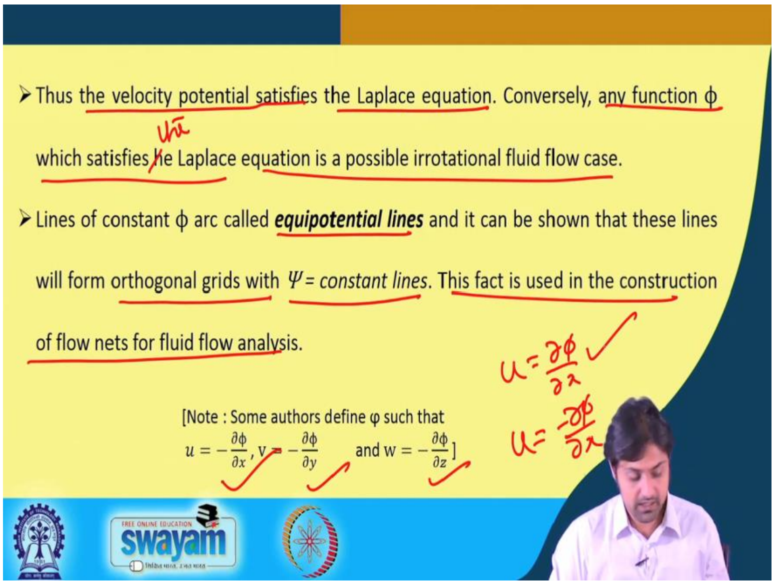

So, we can proceed now. As I said, the velocity potential satisfies the Laplace equation for an incompressible fluid and an irrotational flow. Conversely, any function phi which satisfies the Laplace equation is a possible irrotational fluid flow case. So, this concept will be used in numericals that you are going to solve. So, some any function phi which satisfies the Laplace equation is a possible irrotational fluid flow case. Lines of constant phi velocity potential are called equipotential lines.

This is again an important concept, and it can be shown that these lines will form orthogonal grids with phi is equal to constant lines. This fact is used in construction of the flow nets for fluid flow analysis that we can talk about it more later. There are people can also, I mean, there are some authors which also define instead of + u is equal to del phi / del x they simply prefer u is equal to – del phi del x. It depends on upon lot of convention but any of these formulas is correct. I would ask you to stick with this equation u is equal to del phi / del x.

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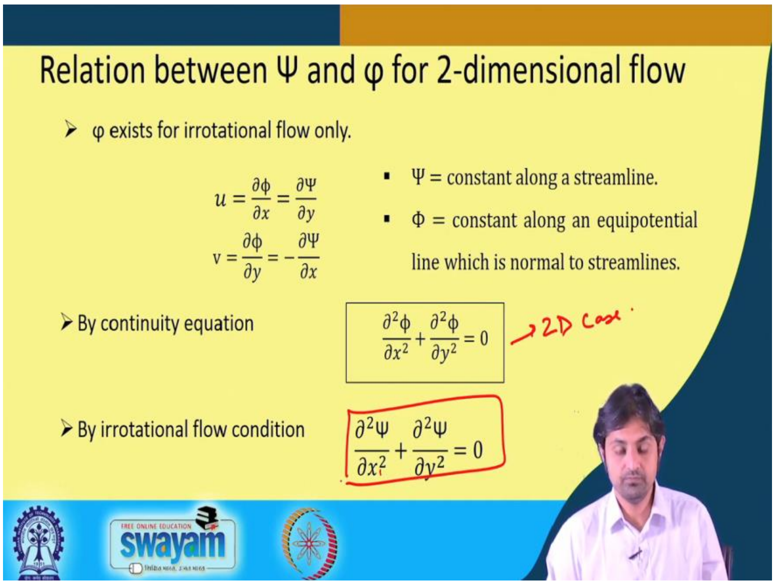

Now, the relationship between phi and psi for 2 dimensional flow. So, as we have seen this phi exist for an irrotational flow only. Then u is equal to del phi / del x velocity potential is equal to del psi / del y. And similarly, v given as, this is potential function del phi / del y and this is = streamlines is - del psi / del x that we have seen before and it is this. So, psi is constant along the stream line and phi is constant along equipotential line which is normal to the streamlines.

By continuity equation what we can write is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

for 2d case. By irrotational flow condition we can write

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = 0$$

this is an important, you see, both satisfying the Laplace equation for irrotational flow condition for continuity equation.

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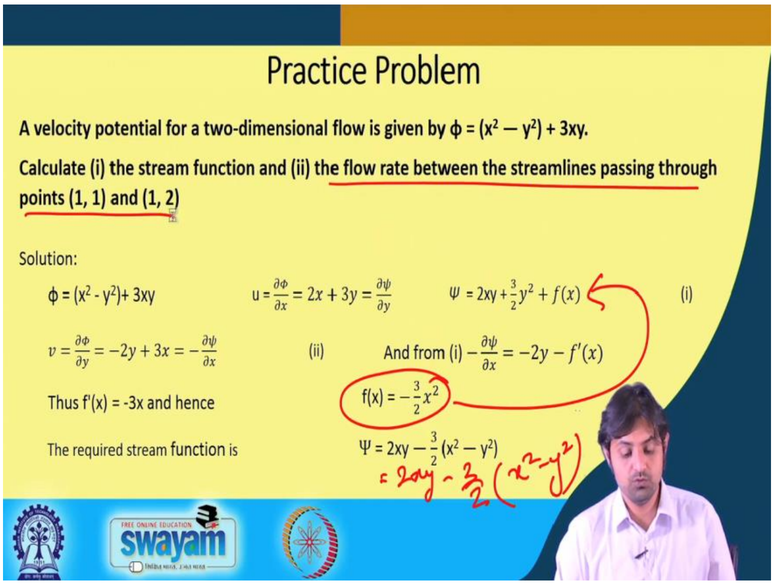

Now, we will see a practice problem. So there is a velocity potential for 2 dimensional flow, which is given by phi is equal to x square - y square + 3 x y. First, we have to calculate the stream function, and then we have to calculate the flow rate between the streamlines passing through the points 1, 1 and 1, 2. So, this question will actually clarify all your practical, if you have any doubt but it is very simple to calculate. So, we can look at the solution. So phi, first we write what is given to us, so, we have written phi = x square - y square + 3 x y.

So, we are going to use one of the equations, u can be written as, del phi / del x this we know and u can also be written as del psi / del y correct. So, from psi we find del phi / del x is equal to this will be 2 x. This will become, I mean, when we start calculating this will become - 0 + 3 y or 2 x + 3 y, but this del phi / del x or u is also equal to del psi / del y. So, we can simply integrate this to obtain psi as it is. So, I will remove this.

So, psi can be integrated to 2 x y + 3 / 2y square very simple integration with respect to y, plus not a constant but anything that is a function of x . Similarly, v is given as del phi / del y. So, we should calculate this. So, I will do this then I will remove del phi / del y is equal to see, x will become 0, del in terms, because there is no y term here. So, this will be 0 - 2 y + 3 x y will give 3 x.

This will become - 2 y + 3 x and this is also equal to - del psi / del x according to the definition, and you integrate this equation and when you integrate this equation, you will be able to solve this for again a psi. So, first let me remove the ink and from one, I mean, this is. So, this is one of the ways that you can solve and then equate, but here what we have done is because psi we already have. So we have calculated from this equation del psi / del x and this came - 2 y - f dash x.

And then in the end we have substituted it with this. Thus, f dash x is equal to, so see, let me just go back. So, from the first equation we have calculated psi , which was a function of x. Using this equation we have calculated del psi / del x we got it - 2 y - f dash x and we have substituted this to this equation. Here, what happened was - 2 y + 3 x was equal to - 2 y - f dash x. Therefore, f dash x became - 3 x. There are 2 different ways the other way was also fine. So, f x, f x is because we obtain f dash x.

So, f x is - 3 / 2 x square. Therefore, the required stream function will be we substitute f of x from here, into 1 to obtain 2 x y + 3 / 2 y square - 3 / 2 x square. So, it can be written as 2 x y - 3 / 2 x square - y square same as this. So, now we are given the point. The second question was we have to we have calculated the stream function, now, the flow rate between the stream lines passing through the points 1, 1 and 1, 2.

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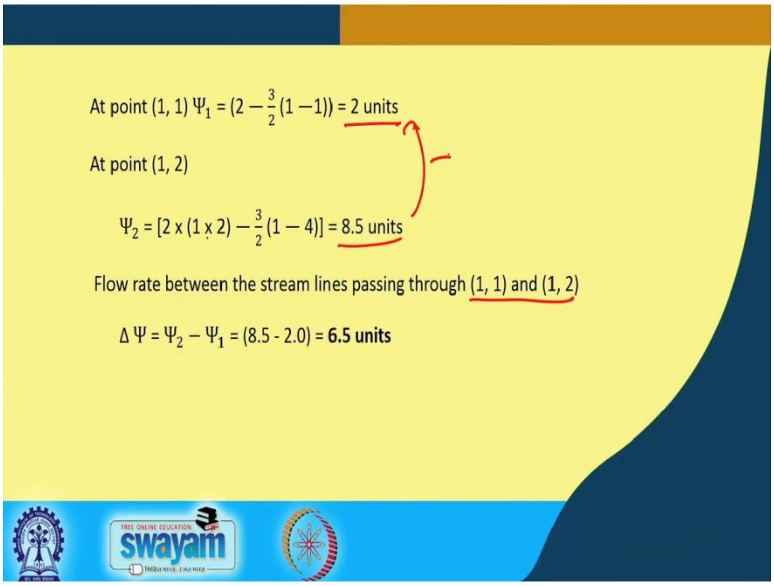

So, at point 1, 1 psi 1 will be just substituting the value of 1 comma 1 will be 2 units, at point 1 comma 2 the psi 2 is going to be substituting 1 comma 2 in the psi that we have obtained gives us 8.5 units and the flow rate between the streamlines passing 1 will be the difference of this and this, and this is going to give us 6.5 units. So, I think with this we have concluded the fluid kinematics part and it is almost time. So, I think we will start with elementary fluid dynamics that is Bernoulli’s equation in our next class. And until then, goodbye. See you next class. Bye bye.