Chapter 22: Rank of a Matrix

Introduction

In linear algebra, matrices are a fundamental mathematical structure used to represent systems of equations, transformations, and data structures. One important concept associated with a matrix is its **rank**, which provides a measure of the matrix's **dimension in terms of linear independence**. The rank tells us about the maximum number of linearly independent row vectors or column vectors in a matrix and plays a critical role in solving linear systems, analyzing consistency of equations, and transforming data.

In civil engineering, matrices and their ranks are used in structural analysis, finite element methods, optimization problems, and more. A proper understanding of rank helps in checking whether a structure is statically determinate or indeterminate, whether a system of forces is solvable, and in analyzing deformation compatibility equations in structures.

22.1 Definition of Rank

The **rank of a matrix** is defined as the maximum number of **linearly independent rows or columns** in the matrix. It is denoted by `rank(A)` for a matrix `A`.

Let A be an $m \times n$ matrix. Then,

$$\text{rank}(A) = \text{maximum number of linearly independent rows or columns of } A$$

A few important points:

- The rank of a matrix is always ≤ the minimum of its number of rows and columns:

  $$\text{rank}(A) \leq min(m, n)$$

- Row rank = Column rank, always.

22.2 Types of Matrix Forms

(a) Row Echelon Form (REF)

A matrix is said to be in **row echelon form** if:

1. All nonzero rows are above any rows of all zeros.
2. The leading coefficient (first non-zero element) of a nonzero row is strictly to the right of the leading coefficient of the row above it.
3. The leading coefficient in any row is 1 (optional for REF).

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Reduced Row Echelon Form (RREF)

A matrix is in reduced row echelon form if:

- It is in REF.
- The leading entry in each nonzero row is 1.
- Each leading 1 is the only non-zero entry in its column.

Example:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

22.3 Elementary Row Operations

To reduce a matrix to REF or RREF, we use **elementary row operations**, which do not change the rank of a matrix. These operations are:

1. **Row swapping**: Interchanging two rows.
2. **Scalar multiplication**: Multiplying a row by a non-zero scalar.
3. **Row addition**: Adding or subtracting a multiple of one row to another row.

These are used in **Gaussian elimination** and **Gauss-Jordan elimination** techniques.

22.4 Methods to Find Rank

Method 1: Echelon Form

1. Reduce the matrix to row echelon form using elementary row operations.
2. Count the number of non-zero rows; this number is the rank of the matrix.

Example: Let

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{bmatrix}$$

Apply row operations:

- $R_2 \rightarrow R_2 - 2R_1$
- $R_3 \rightarrow R_3 - 3R_1$

$$\Rightarrow\begin{bmatrix}1 & 2 & 3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Only one non-zero row remains, so **rank = 1**.

Method 2: Using Minors

1. Find the largest order of a **non-zero determinant of a square submatrix**.
2. The order of the highest such non-zero determinant is the rank.

Example: Let

$$B=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}$$

The determinant of the full 3×3 matrix is 0 (since rows are linearly dependent).
Now check 2×2 minors. Take:

$$det\begin{bmatrix}1 & 2\\ 4 & 5\end{bmatrix}=(1)(5)-(4)(2)=5-8=-3\neq 0$$

So **rank = 2**

22.5 Rank of Special Matrices

(a) Zero Matrix

A zero matrix (all elements 0) has **rank 0**.

(b) Identity Matrix I_n

The identity matrix of order n has **rank n** because all rows (and columns) are linearly independent.

(c) Diagonal Matrix

A diagonal matrix with k non-zero diagonal elements has **rank = k**.

(d) Upper or Lower Triangular Matrix

Its rank is equal to the number of non-zero rows, as they are already in echelon form.

22.6 Applications of Rank in Civil Engineering

1. **Solving Linear Systems** Rank determines the **consistency** of linear systems:
    - If $\text{rank}(A) = \text{rank}([A \vee B])$, the system is consistent.
    - If $\text{rank}(A) = \text{rank}([A \vee B]) = n$, the system has a unique solution.
    - If $\text{rank}(A) = \text{rank}([A \vee B]) < n$, the system has infinitely many solutions.
    - If $\text{rank}(A) \neq \text{rank}([A \vee B])$, the system has no solution.
2. **Structural Analysis**
    - Rank is used to identify whether a structure is determinate or indeterminate.
    - The rank of stiffness or flexibility matrices helps determine solvability of nodal displacements and internal forces.
3. **Finite Element Method (FEM)**
    - FEM involves assembling global stiffness matrices whose rank determines whether a system can be solved or if constraints are insufficient.
4. **Data Interpretation and Optimization**

- o In transportation planning and scheduling, rank helps in solving constraint optimization problems.

22.7 Consistency of a Linear System: Rank-Based Approach

Let us consider a linear system of equations:

$$AX = B$$

Where:

- A is the coefficient matrix,
- X is the column vector of unknowns, and
- B is the column vector of constants.

The **augmented matrix** is $[A \vee B]$. The consistency and solution nature of this system depends on comparing the ranks of A and $[A \vee B]$.

22.7.1 Theorem: Rouché–Capelli Theorem

A system of linear equations is **consistent** (i.e., has at least one solution) **if and only if**

$$\text{rank}(A) = \text{rank}([A \vee B])$$

If the system is consistent:

- **Unique solution** if $\text{rank}(A) = \text{number of variables}$
- **Infinitely many solutions** if $\text{rank}(A) < \text{number of variables}$

If the system is **inconsistent**:

$$\text{rank}(A) \neq \text{rank}([A \vee B])$$

Example 1: Consistent System

Solve:

$$x + y + z = 6 \\   x + 2y + 3z = 14 \\   2x + 3y + 4z = 20$$

Coefficient matrix A:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} B = \begin{bmatrix} 6 \\ 14 \\ 20 \end{bmatrix}$$

Form augmented matrix and apply row operations:

$$[A \vee B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 2 & 3 & 4 & 20 \end{bmatrix} \Rightarrow \text{Reduce to REF}$$

After reduction:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(A) = \text{rank}([A \vee B]) = 2 < 3$$

⇒ **Infinitely many solutions**
