Chapter 32: Basis of Eigenvectors

Introduction

In the study of linear algebra, especially for systems that arise in civil engineering—like structural analysis, fluid mechanics, or stress-strain problems—understanding **eigenvalues** and **eigenvectors** is essential. Once the eigenvectors of a matrix are determined, they can form a basis for vector spaces associated with the matrix, especially the eigenspaces. This chapter focuses on how eigenvectors form a basis, how to construct such bases, and the implications in engineering applications such as analyzing structural modes or vibrations.

32.1 Eigenvectors and Eigenspaces

Let A be an $n \times n$ matrix. If there exists a non-zero vector $v \in R^n$ and a scalar $\lambda \in R$ such that:

$$Av = \lambda v$$

then λ is an **eigenvalue** of A, and v is an **eigenvector** corresponding to λ.

Eigenspace

The **eigenspace** corresponding to an eigenvalue λ is the set:

$$E_\lambda = \{ v \in R^n : Av = \lambda v \}$$

It can also be expressed as:

$$E_\lambda = \text{Null}(A - \lambda I)$$

which is a **subspace** of R^n. The **dimension** of E_λ is called the **geometric multiplicity** of λ.

32.2 Basis of an Eigenspace

To understand the **basis of eigenvectors**, we focus on finding a **basis** for each eigenspace E_λ.

Let's suppose:

- A is an $n \times n$ matrix
- λ is an eigenvalue of A
- We solve $(A - \lambda I)v = 0$

The **solutions** to this homogeneous system form a **vector space**, and the vectors that span this space are the eigenvectors corresponding to λ.

A **basis** of the eigenspace is a **linearly independent set of eigenvectors** that spans the entire eigenspace.

32.3 Steps to Find Basis of Eigenvectors

Step 1: Find the Eigenvalues

Solve the **characteristic equation**:

$$det(A - \lambda I) = 0$$

This will give the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$.

Step 2: Find Eigenspaces

For each eigenvalue λ_i, solve the equation:

$$(A - \lambda_i I)v = 0$$

This gives the **null space** of $(A - \lambda_i I)$, which is the eigenspace E_{λ_i}.

Step 3: Determine the Basis

From the general solution of $(A - \lambda_i I)v = 0$, extract a **set of linearly independent vectors** that span E_{λ_i}.

These vectors form the **basis of eigenvectors** for λ_i.

32.4 Algebraic and Geometric Multiplicity

For each eigenvalue λ:

- **Algebraic multiplicity (AM):** Number of times λ appears as a root of the characteristic polynomial.

- **Geometric multiplicity (GM):** Dimension of the eigenspace E_λ, i.e., number of linearly independent eigenvectors for λ.

Important Property:

$$1 \leq \text{GM}(\lambda) \leq \text{AM}(\lambda)$$

If $\text{GM} = \text{AM}$ for all eigenvalues, then the matrix is **diagonalizable**.

32.5 Example

Let:

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

Step 1: Characteristic Equation

$$det(A - \lambda I) = det \begin{bmatrix} 4-\lambda & 1 \\ 0 & 4-\lambda \end{bmatrix} = ¿$$

So, $\lambda = 4$ is a **repeated eigenvalue** (AM = 2).

Step 2: Eigenspace

Solve $(A - 4I)v = 0$:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow y = 0$$

So, $x \in R$, and the eigenspace is:

$$E_4 = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Only **one** linearly independent eigenvector ⇒ GM = 1 < AM = 2 ⇒ Matrix is **not diagonalizable**.

Basis of eigenvectors for $\lambda = 4$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

32.6 Application in Civil Engineering

Understanding the basis of eigenvectors is critical in:

- **Modal analysis** of structures where each mode shape is an eigenvector
- **Dynamic analysis** of buildings subjected to seismic waves
- **Principal stress directions** in stress-strain analysis
- **Stability analysis** in frames and trusses

When matrices representing systems are symmetric (as in stiffness matrices), the eigenvectors are **orthogonal** and form an **orthonormal basis**, simplifying the analysis of structural vibrations.

32.7 Diagonalization and Basis of Eigenvectors

If a matrix A has n linearly independent eigenvectors, then:

$$A = P D P^{-1}$$

Where:

- P is a matrix with columns as eigenvectors (a **basis** of eigenvectors),
- D is a diagonal matrix with corresponding eigenvalues.

This is **diagonalization**, and it's only possible if eigenvectors form a complete basis for R^n.

32.8 Orthogonal Basis (for Symmetric Matrices)

For **symmetric matrices**, the **Spectral Theorem** states:

- All eigenvalues are **real**
- Eigenvectors corresponding to distinct eigenvalues are **orthogonal**

Thus, for symmetric matrices A:

- One can form an **orthonormal basis** from the eigenvectors.
- This basis simplifies projections, decompositions, and principal component analysis (PCA) in structural engineering.

32.9 Summary of Key Concepts

- Eigenspaces are null spaces of $(A-\lambda I)$
- Basis of eigenvectors is a linearly independent set spanning the eigenspace
- If eigenvectors form a basis for R^n, matrix is diagonalizable
- For symmetric matrices, eigenvectors can form an **orthonormal** basis

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32.10 Extended Example: 3×3 Matrix

Let us consider the matrix:

$$A=\begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 4\\ 0 & 4 & 3\end{bmatrix}$$

Step 1: Characteristic Polynomial

$$det(A-\lambda I)=det\begin{bmatrix}2-\lambda & 0 & 0\\ 0 & 3-\lambda & 4\\ 0 & 4 & 3-\lambda\end{bmatrix}$$

Expanding along the first row:

$$(2-\lambda)\cdot det\begin{bmatrix}3-\lambda & 4\\ 4 & 3-\lambda\end{bmatrix}$$

$$¿(2-\lambda)¿$$

$$¿(2-\lambda)(7-\lambda)(-1-\lambda)$$

So, eigenvalues are:

$$\lambda_1=2,\lambda_2=7,\lambda_3=-1$$

Step 2: Find Eigenvectors

For $\lambda=2$***:***

$$A-2I=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 4\\ 0 & 4 & 1\end{bmatrix}\Rightarrow(A-2I)v=0$$

Solve:

$$\begin{cases} v_2 + 4v_3 = 0 \\ 4v_2 + v_3 = 0 \end{cases} \Rightarrow v_2 = -4v_3, v_3 = -4v_2$$

This implies:

$$v_2 = 0, v_3 = 0, v_1 = \text{free} \Rightarrow v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus, basis of eigenvectors for $\lambda = 2$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(Similarly solve for $\lambda = 7$ and $\lambda = -1$)

32.11 Complex Eigenvalues and Basis

In some real matrices, eigenvalues may be **complex**. For example:

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Characteristic polynomial:

$$det(B - \lambda I) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

For complex eigenvalues, we find complex eigenvectors. However, these may not form a basis over R^n, only over C^n.

In **engineering** applications (e.g., oscillations), these are interpreted using **Euler's formula** to convert to real-valued trigonometric solutions.

32.12 Diagonalizability and Basis of Eigenvectors

Recall: A matrix $A \in R^{n \times n}$ is diagonalizable if and only if it has n linearly independent eigenvectors.

Key Conditions for Diagonalizability

- A has n **distinct** eigenvalues ⇒ **Always** diagonalizable.

- If any eigenvalue has **geometric multiplicity < algebraic multiplicity**, then A is **not** diagonalizable.
- **Symmetric** matrices are **always diagonalizable** with real eigenvalues and orthogonal eigenvectors.

Diagonalization Formula

$$A = P D P^{-1}$$

- Columns of P: Basis of eigenvectors
- D: Diagonal matrix of eigenvalues

32.13 Role in Structural Dynamics

In civil engineering, particularly in **structural dynamics** and **vibrations**, the stiffness matrix K and mass matrix M define the equation:

$$K x = \lambda M x$$

This is a **generalized eigenvalue problem**.

- λ: Square of natural frequencies ω^2
- x: Mode shapes (eigenvectors)

These mode shapes form a **basis** of independent vibration directions.

Properties:

- Eigenvectors (mode shapes) are **orthogonal** under the mass or stiffness metric.
- Used in **modal analysis** to decouple the system into independent SDOF (single degree of freedom) systems.

32.14 Summary Table of Concepts

Concept	Definition
Eigenvector	Non-zero vector v satisfying $A v = \lambda v$
Eigenspace	Null space of $(A - \lambda I)$, a vector subspace
Basis of Eigenvectors	Linearly independent

Concept	Definition
	eigenvectors spanning an eigenspace
Geometric Multiplicity	Dimension of eigenspace
Algebraic Multiplicity	Number of times eigenvalue occurs in characteristic polynomial
Diagonalizable	Matrix with n linearly independent eigenvectors
Orthonormal Basis	Eigenvectors that are orthogonal and of unit length (for symmetric matrices)