

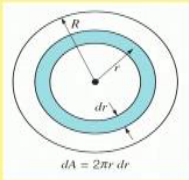
Hydraulic Engineering
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Lecture - 40
Pipe Flow (Contd.,)

Welcome back student to yet another lecture of the pipe flow.

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Newton's 2nd law



Flow rate: $Q = \int u dA = \int_0^{D/2} u(r) 2\pi r dr = \frac{\pi D^4 \Delta p}{128 \mu l}$ Eq. 4

Poiseuille's Law $Q = \frac{\pi D^4 \Delta p}{128 \mu l}$

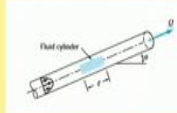
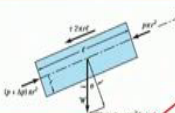
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In the last lecture we derived the Poiseuille's law. The Poiseuille's law is $Q = \frac{\pi D^4 \Delta p}{128 \mu l} Q$, where l is length of the pipe and Δp is the pressure drop. So, we have been able to relate the pressure drop with the discharge. So, proceeding forward,

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Newton's 2nd law

• if gravity is present, it can be added to the pressure:

$$\frac{\Delta p - \rho g l \sin \theta}{l} = \frac{2\tau}{r}$$

$$V = \frac{(\Delta p - \rho g l \sin \theta) D^2}{32 \mu l}$$

$$Q = \frac{\pi (\Delta p - \rho g l \sin \theta) D^4}{128 \mu l}$$

Eq. 5

Eq. 6

Eq. 7

Generalized

We will now see what if the gravity was also present and that if it can be added to the pressure. So, it is, earlier our pipe was horizontal, there was no gravity. But right now, the equations of motion will change. Because then there will be another, you know, this $w \sin \theta$, that is, $\pi r^2 l \sin \theta$, this is not multiplied by γ , $\gamma \pi r^2 l \sin \theta$ will be another component that will be acting in this direction.

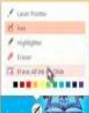
Because of which our analysis will change a bit but the what happens is, instead of Δp , we are going to get components like, minus $\rho g l \sin \theta$, in the equation number 1. When we are going to calculate the velocity, minus $\rho g l \sin \theta$ here and the Poiseuille's law will change, where minus $\rho g l \sin \theta$. If, you want to verify, see, this is valid for all the angles. So, if you put $\theta = 0$, you see, this term will vanish, this term will vanish and this term, because $\sin 0 = 0$.

It will be same as what we have derived before. So, you can also try to remember these equations, at least Poiseuille's law you remember, that is quite important, with or without the, I mean, the change in angle, you know, if the gravity is there or not with the sloping pipe. So, these are the generalized equation.

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Class Question

- An oil with a viscosity of $0.40 \text{ N}\cdot\text{s}/\text{m}^2$ and density of $900 \text{ kg}/\text{m}^3$ flows in a pipe of diameter 0.002 m . What pressure drop, is needed to produce a flow rate of $2.0 \times 10^{-5} \text{ m}^3/\text{s}$ if the pipe is horizontal with and $x_1=0$ and $x_2=10 \text{ m}$.
- How steep a hill, must the pipe be on if the oil is to flow through the pipe at the same rate as above, but with $\Delta p=0$.
- For the conditions of part b, if $p_1=200 \text{ kPa}$ what is the pressure at section $x_3=5 \text{ m}$ where x is measured along the pipe




So, now, we are going to solve a question, which says that an oil with viscosity of $0.4 \text{ Newton second per meter square}$ and density of $900 \text{ kilogram per meter cube}$, flows in a pipe of diameter 0.02 meter . What pressure drop, is needed to produce a flow rate of $2 \text{ into } 10 \text{ to the power } -5 \text{ meter cube per second}$ if the pipe is horizontal with $x_1 = 0$ and $x_2 = 10 \text{ meter}$? That is the first question.

Another question is, how steep a hill must the pipe be on if the oil is to flow through the pipe at the same rate as above with $\Delta p = 0$. And the third, for the conditions of part b, this one, if $p_1 = 200 \text{ kilopascal}$ what is the pressure at section $x_3 = 5 \text{ meter}$, where x is measured along the pipe. So, this is the question that involves the use of Poiseuille's law and also involves the last equations that we have seen, what happens if the gravity was present. So, to solve this, we will solve this in a white screen.

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Solution 2

a) If the Reynolds number is less than 2100 the flow is laminar and the equations derived before are valid.

Since the average velocity is

$$V = \frac{Q}{A} = \frac{2 \times 10^{-5}}{\frac{\pi}{4} (0.020)^2} = 0.0637 \text{ m/s}$$

$Re = \frac{\rho V D}{\mu} = 2.87$ $Re < 2100 \Rightarrow$ Hence the flow is laminar and $L = x_2 - x_1 = 10 \text{ m}$ → Poiseuille's law

$\Delta p = p_1 - p_2 = \frac{128 \mu L Q}{\pi D^4}$

The solution 2, the first one. So, if the Reynolds number is less than 2100 the flow is laminar and the equations derived before are valid. Since, the average velocity is $V = Q/A$. So, Q is 2, into 10 to the power -5 and area is $\pi/4$, into 0.020 whole square and this comes to be 0.0637 meters per second. And how do we write the Reynolds number? $\rho V D / \mu$ and on putting this you will get Reynolds number, this velocity.

We are going to get, Reynolds number of 2.87. This Reynolds number is less than 2100 which implies, hence, the flow is laminar and $L = x_2 - x_1 = 10$ meter. So, first we found out the velocity. Checked that using the Reynold number, if it is laminar or not. Yes, it was laminar and the pressure drop was $p_1 - p_2 = 128$, using the Poiseuille's law. So, we are now going to substitute that.

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Handwritten calculations on a whiteboard:

Left side:

$$\Delta p = p_1 - p_2 = \frac{128 \mu l Q}{\pi D^4}$$

$$\Delta p = \frac{128 \times 0.40 \times 10 \times 2 \times 10^{-5}}{\pi \times (0.020)^4}$$

$$\Delta p = 20371 \text{ N/m}^2 = 20.4 \text{ kPa}$$

Below this, it says: "b) If the pipe is on a hill of angle θ such that $\Delta p = p_1 - p_2 = 0$ we use modified Poiseuille's law"

Right side:

$$\sin \theta = - \frac{128 \mu Q}{\pi \rho g D^4}$$

$$\sin \theta = \frac{-128 \times 0.40 \times 2 \times 10^{-5}}{\pi \times 900 \times 9.81 \times (0.020)^4}$$

A box contains the result: $\theta = -13.34^\circ$

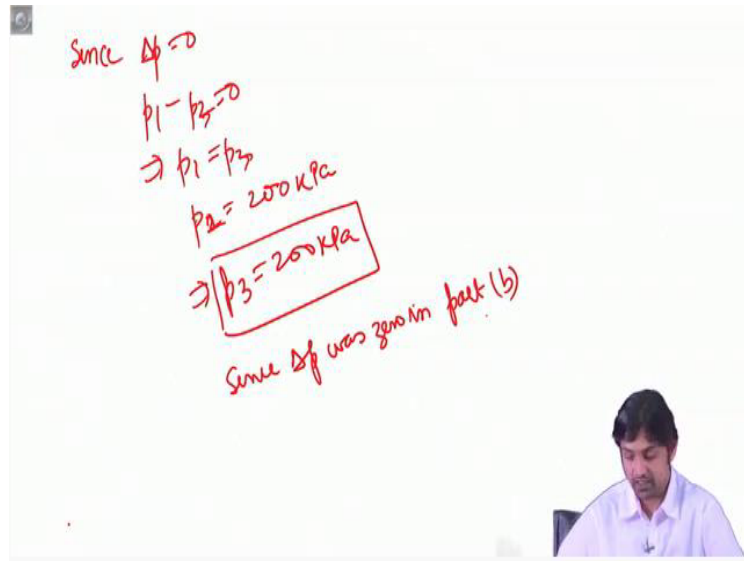
Below the box, it says: "c) With $p_1 = p_2$, the length of pipe l , does not appear in flow rate equation. This is a statement of the fact that for such cases the pressure is constant all along the pipe"

So, delta p, I will rewrite the equation, is equal to $p_1 - p_2 = 128 \mu l Q / \pi D$ to the power 4. So, delta p is going to be 128μ is 0.40, into 10 into 2 into 10 to the power - 5 / π into 10 to the power 4. So, pressure drop is going to be 20371 Newton per meter square or 20.4 kilopascal. This is the pressure drop that we found out using the Poiseuille's law. The second part says, if the pipe is on a hill of angle theta, such that, the pressure drop $p_1 - p_2 = 0$.

So, for this, we use modified Poiseuille's law and we can write, because $\Delta p = 0$, so we can use $\sin \theta$ will be $-128 \mu Q / \pi \rho g$ into D to the power 4. So, $\sin \theta$ is going to be -128 this is 0.40, into 2 into 10 to the power -5 into π into 900 into 9.81 into 0.020 to the power 4 and theta is going to be approximately -13.34 degrees. So, in part a, we use simple Poiseuille's law. In the second one, we use the modified Poiseuille's law.

Now, the third part says that with $p_1 = p_2$, the length of pipe l does not appear in the flow rate equation. So, this is a statement of the fact that for such cases, the pressure is all along the pipe. See, for part, b we said there was no pressure drop, that is correct. So, that means the pressure is constant all along the pipe.

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Since, $\Delta p = 0$, $p_1 - p_2 = 0$. So, p_1 is equal to p_2 . And p_2 is equal to how much? Sorry, $p_1 = 200$ kilopascal, sorry, or yes, p or 3, sorry. Therefore, p_3 is also going to be 200 kilopascal, since Δp was 0, in part b, very simple, because there was no pressure drop per unit length. So, the pressure would be the same as, what is at 0. So, we finish this one here. So, in the previous question, we have seen the application of Poiseuille's law and also the modified Poiseuille's law, in presence of the gravity.

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Basic mathematics (Revision)

- Del Operator: $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$
- Laplacian Operator: $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
- Gradient: $\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right)$




With this point, we move forward to our second way of finding the derived, for finding the laminar flow in the pipes, for fully developed flow. But before that we should revise some basic

mathematics, for revision and these concerns our mathematical operator. For example, there is an operator called del operator, which every one of you know, it is

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

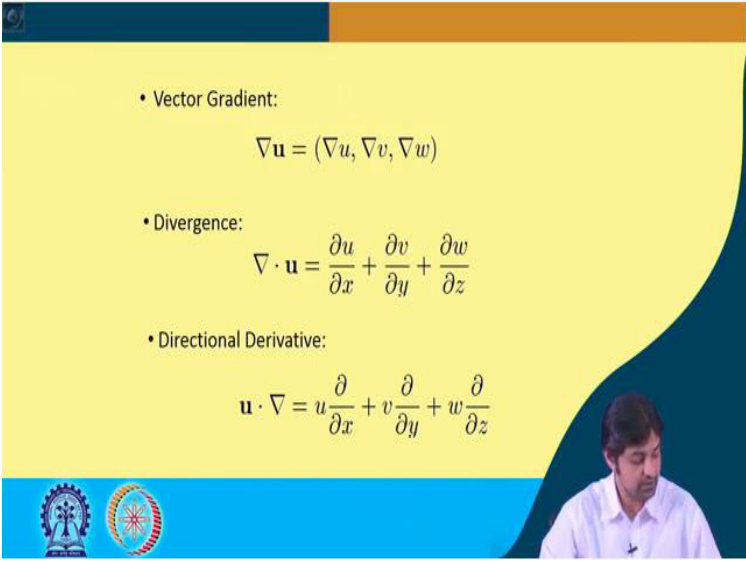
. Now, the Laplacian operator is

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

. Gradient operator, it is

$$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right)$$

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- Vector Gradient:

$$\nabla \mathbf{u} = (\nabla u, \nabla v, \nabla w)$$
- Divergence:

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$
- Directional Derivative:

$$\mathbf{u} \cdot \nabla = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$$

There is a vector gradient, that is,

$$\nabla \mathbf{u} = (\nabla u, \nabla v, \nabla w)$$

. There is a divergence operator, that is,

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

. These are all basic mathematical equations that you know from before. There is one directional derivative that

$$\mathbf{u} \cdot \nabla = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$$

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Navier-Stokes equation applied

- General motion of an incompressible Newtonian fluid is governed by the continuity equation conservation of mass, is written as :

$$\nabla \cdot \mathbf{u} = 0$$
→ Continuity Equation
- Momentum equation: $\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\frac{\nabla p}{\rho} + \mathbf{g} + \nu \nabla^2 \mathbf{V}$
→ Momentum Equation / Navier Stokes Equation Eq. 8

For steady, fully developed flow in a pipe, the velocity contains only an axial component, which is a function of only the radial coordinate $[V=u(r)]$. For such conditions, the left-hand side of the momentum Eqn. becomes zero. This is equivalent to saying that the fluid experiences no acceleration as it flows along. The same constraint was used in the previous section when considering $F = ma$ for the fluid cylinder.



So, with this thing we are proceeding ahead for derivation of the laminar flow through pipes, using the Navier-Stokes equation. So, the general motion of an incompressible Newtonian fluid is governed by the continuity equation and the conservation of mass. This continuity equation is written as, $\nabla \cdot \mathbf{u} = 0$, this is continuity equation. Whereas, the momentum equation is written as, this or Navier-Stokes equation and this is equation number 8.

So, you should remember that for steady fully developed flow in the pipe the velocity contains only an axial component, which is the function of only the radial coordinates r , that we have talked in the beginning of this chapter itself. So, for such conditions, the left hand side of the momentum equation becomes 0, because it is steady. This is equivalent to saying that the fluid experiences no acceleration as it flows along.

So, convective acceleration and everything is 0. This is acceleration, local acceleration, there is convective acceleration. The same constraint was used in the previous section, when we considered $F = ma$, for fluid cylinders. So, left side of the Navier-Stokes equation becomes 0, because there is no acceleration in the flow.

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Thus, with $\underline{g} = -g\hat{k}$ the Navier Stokes equations become

$$\nabla \cdot \underline{V} = 0$$

$$\underline{\nabla} p + \rho \underline{g} \hat{k} = \mu \nabla^2 \underline{V}$$

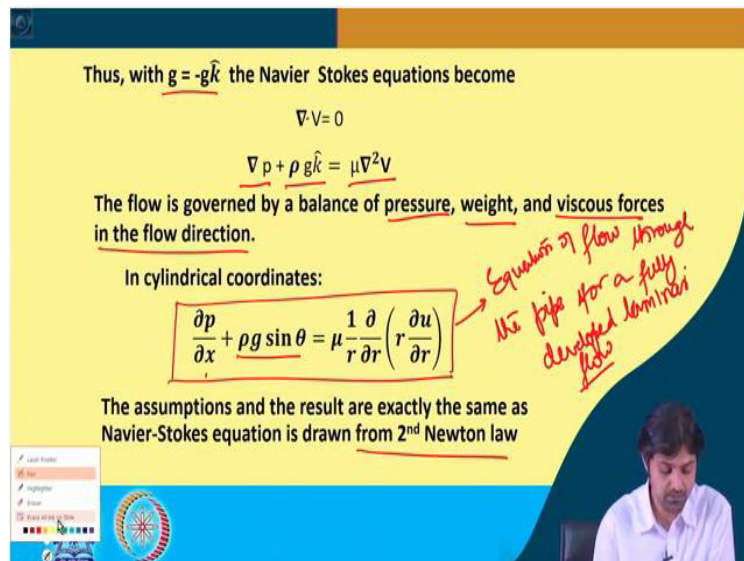
The flow is governed by a balance of pressure, weight, and viscous forces in the flow direction.

In cylindrical coordinates:

$$\frac{\partial p}{\partial x} + \rho g \sin \theta = \mu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right)$$

Equation of flow through the pipe for a fully developed laminar flow

The assumptions and the result are exactly the same as Navier-Stokes equation is drawn from 2nd Newton law



Therefore, if we write, with g as, minus g , in terms of vector, the Navier-Stokes equation, so the continuity equation will be $\nabla \cdot \underline{v} = 0$ and the right hand side can be written as Δp , $\Delta p + \rho g \hat{k} + \mu \nabla^2 \underline{V}$. Now, the flow is governed by the balance of pressure, weight and viscous forces in flow direction. In cylindrical coordinates, we can simply write, so ∇p is $\frac{\partial p}{\partial x} + \rho g \sin \theta$. Whereas, μ in radial direction $\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right)$.

So, this is the using the Navier-Stokes equation, equation of flow through the pipe for a fully developed laminar flow. So, here are the assumptions and the results are exactly same as the Navier-Stokes equation, which is derived from the Newton second law, just written in a more differential form.

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Dimensional analysis

$$\Delta p = F(V, l, D, \mu)$$

$$\frac{D \Delta p}{\mu V} = \phi\left(\frac{l}{D}\right) \rightarrow$$

assuming pressure drop proportional to the length:

$$\frac{D \Delta p}{\mu V} = \frac{Cl}{D} \Rightarrow \frac{\Delta p}{l} = \frac{C \mu V}{D^2}$$

$$Q = AV = \frac{(\pi/4C) \Delta p D^4}{\mu l}$$

Eq. 9

$\frac{D \Delta p}{\mu V} = C \cdot \frac{l}{D}$
 $\Rightarrow \frac{\Delta p}{l} = \frac{C \mu V}{D^2}$
 $Q = AV = \frac{\pi \cdot C \mu V D^4}{4 \mu l}$

Now, the third way I said was dimensional analysis, from your dimensional analysis module. You remember that pressure is a function of V, l, D, and Mu. So, if we take delta p/l as a function of V, D, Mu or let us take, K is how much, 5, r is, so 1, 2, 3, 4, 5. So, K is 5 and r is 3. So, therefore number of dimensionless terms will be K-r, that is, 2 dimensionless term, so, pi 1 and pi 2. We get this using dimensional analysis.

So, using dimensional analysis we get this result. Now, assuming that the pressure drop is proportional to the length, so we say that the D delta p/Mu V is directly proportional to the length so it becomes C into l/D. So, from here, because you might not be, so D delta p/Mu V is some function of l/D, but we say proportional to the length.

So, we write, constant C into l/D, this equation. Therefore, we can write, delta p/l, l we bring this side, other parameters we take that side, it becomes C Mu V/D square or Q = area into velocity. So, area is pi/4 D square. Therefore, we will get, pi/4 into C delta p D to the power 4/Mu l, as this equation. Now, you see, there is something, some constant called C.

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Darcy friction factor (f)

- Rewriting Poiseuille's law

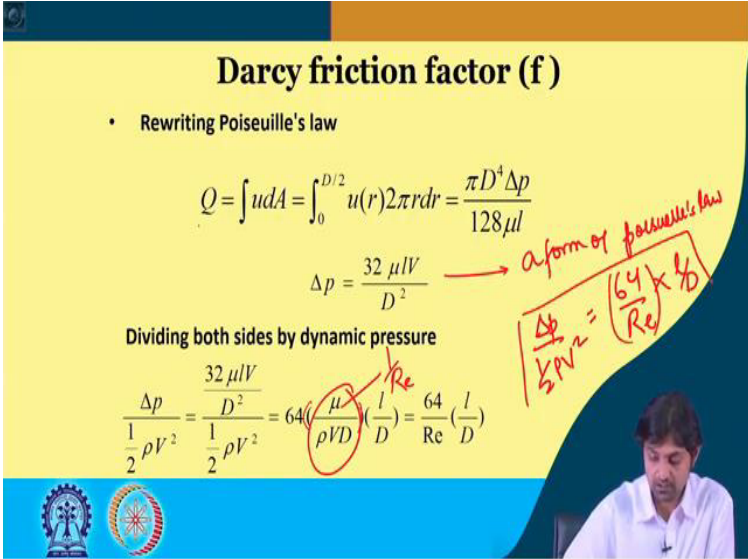
$$Q = \int u dA = \int_0^{D/2} u(r) 2\pi r dr = \frac{\pi D^4 \Delta p}{128 \mu l}$$

$$\Delta p = \frac{32 \mu l V}{D^2}$$

Dividing both sides by dynamic pressure

$$\frac{\Delta p}{\frac{1}{2} \rho V^2} = \frac{32 \mu l V}{D^2} \cdot \frac{2}{\rho V^2} = 64 \left(\frac{\mu}{\rho V D} \right) \left(\frac{l}{D} \right) = \frac{64}{Re} \left(\frac{l}{D} \right)$$

a form of Poiseuille's law
 $\frac{\Delta p}{\frac{1}{2} \rho V^2} = \frac{64}{Re} \times \frac{l}{D}$



We are going to look into something here that is called Darcy's friction factor. If, we rewrite the Poiseuille's law, Poiseuille's law was $\pi D^4 \Delta p / 128 \mu l$. Therefore, Δp could be written as, $32 \mu l V / D^2$, in terms of velocity. Now, if we divide both sides by this side, so this is in terms of Q and this is in terms of V . So, this is also a form of Poiseuille's law.

If, you divide both sides by dynamic pressure, like $\frac{1}{2} \rho V^2$, so $\Delta p / \frac{1}{2} \rho V^2$ will be $32 \mu l V / D^2$ divided by $\frac{1}{2} \rho V^2$. So, this will be, this half becomes, goes up and become 64 and this we can take out, $\mu / \rho V D$ into $1/D$ and this $\mu / \rho V D$ is $1/Re$. So, basically, we can write, $\Delta p / \frac{1}{2} \rho V^2 = 64 / Re \times l/D$. So, you see, we would see later, that this $64/Re$ is termed as Darcy friction factor f . Darcy means this is for the laminar flow.

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Darcy friction factor (f)

- Often written as

$$\Delta p = f \left(\frac{l}{D} \right) \frac{\rho V^2}{2} \quad \text{Eq. 10}$$

Where f is called Darcy friction factor.

f for laminar fully developed pipe flow is given by

$f = \frac{64}{Re}$

→ Darcy friction factor (f) for fully developed laminar flow



So, this is often written as, $\Delta p = f \text{ into } l/D \rho V^2/2$, with this equation number 10 analogy, we say that Darcy friction f for laminar fully developed pipe flow is given by $64/Re$ and this is Darcy friction factor f for fully developed laminar flow. And this is you must remember this.

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Darcy friction factor (f)

- Often written as

$$\Delta p = f \left(\frac{l}{D} \right) \frac{\rho V^2}{2} \quad \text{Eq. 10}$$

Where f is called Darcy friction factor.

f for laminar fully developed pipe flow is given by

$$f = \frac{64}{Re} \quad \text{Eq. 11}$$

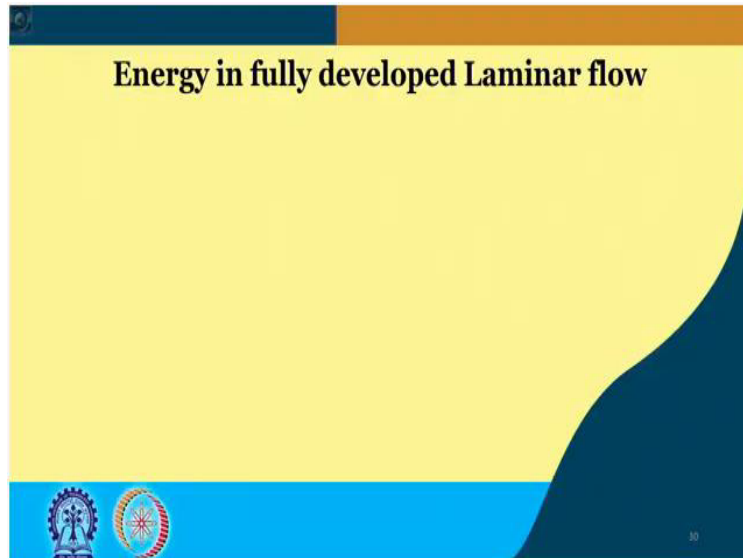
In terms of wall shear stress Using Eq. 3

$$f = \frac{8\tau_w}{\rho V^2} \quad \text{Eq. 12}$$

This is equation number 11. So, now in terms of wall shear stress, if we use equation number 3, f can be written as, so equation if we use this $f l/D$ into $\rho V^2/2$. If we if in terms of wall shear stress using equation 3 we will get $f = 8 \tau_w / \rho V^2$. This is quite simple. Do it as your homework. If you have any problems, you can ask that to me in your, in the forum and this is equation number 12. It is just a matter of equating to obtain this, in terms of wall shear stress.

So, I think this is a nice point to stop this lecture and I will see you in the next lecture where we are going to continue the topic of energy in fully developed laminar flows.

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Thank you so much for listening and I will see you in the next class.