Chapter 12

KINEMATIC INDETERMINANCY; STIFFNESS METHOD

12.1 Introduction

12.1.1 Stiffness vs Flexibility

1 There are two classes of structural analysis methods, Table 12.1:

Flexibility: where the primary unknown is a force, where equations of equilibrium are the starting point, static indeterminancy occurs if there are more unknowns than equations, and displacements of the entire structure (usually from virtual work) are used to write an equation of compatibility of displacements in order to solve for the redundant forces.

Stiffness: method is the counterpart of the flexibility one. Primary unknowns are displacements, and we start from expressions for the forces written in terms of the displacements (at the element level) and then apply the equations of equilibrium. The structure is considered to be *kinematically indeterminate* to the nth degree where n is the total number of independent displacements. From the displacements, we then compute the internal forces.

	Flexibility	**Stiffness**
Primary Variable (d.o.f.)	Forces	Displacements
Indeterminancy	Static	Kinematic
Force-Displacement	Displacement(Force)/Structure	Force(Displacement)/Element
Governing Relations	Compatibility of displacement	Equilibrium
Methods of analysis	“Consistent Deformation”	Slope Deflection; Moment Distribution

Table 12.1: Stiffness vs Flexibility Methods

2 In the flexibility method, we started by releasing as many redundant forces as possible in order to render the structure *statically determinate*, and this made it quite flexible. We then

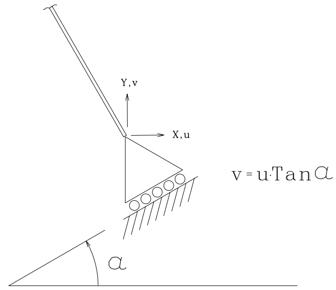


Figure 12.2: Independent Displacements

Type		Node 1	Node 2
1 Dimensional			
Beam	$\{\mathbf{p}\}$	F_{y1}, M_{z2}	F_{y3}, M_{z4}
	$\{\boldsymbol{\delta}\}$	v_1, θ_2	v_3, θ_4
2 Dimensional			
Truss	$\{\mathbf{p}\}$	F_{x1}	F_{x2}
	$\{\boldsymbol{\delta}\}$	u_1	u_2
Frame	$\{\mathbf{p}\}$	F_{x1}, F_{y2}, M_{z3}	F_{x4}, F_{y5}, M_{z6}
	$\{\boldsymbol{\delta}\}$	u_1, v_2, θ_3	u_4, v_5, θ_6
3 Dimensional			
Truss	$\{\mathbf{p}\}$	F_{x1},	F_{x2}
	$\{\boldsymbol{\delta}\}$	u_1,	u_2
Frame	$\{\mathbf{p}\}$	F_{x1}, F_{y2}, F_{y3}, T_{x4} M_{y5}, M_{z6}	F_{x7}, F_{y8}, F_{y9}, T_{x10} M_{y11}, M_{z12}
	$\{\boldsymbol{\delta}\}$	u_1, v_2, w_3, θ_4, θ_5 θ_6	u_7, v_8, w_9, θ_{10}, θ_{11} θ_{12}

Table 12.2: Degrees of Freedom of Different Structure Types Systems

12.2.1 Methods of Analysis

12 There are three methods for the stiffness based analysis of a structure

Slope Deflection: (Mohr, 1892) Which results in a system of n linear equations with n unknowns, where n is the degree of kinematic indeterminancy (i.e. total number of independent displacements/rotation).

Moment Distribution: (Cross, 1930) which is an iterative method to solve for the n displacements and corresponding internal forces in flexural structures.

Direct Stiffness method: ($\tilde{1}960$) which is a formal statement of the stiffness method and cast in matrix form is by far the most powerful method of structural analysis.

The first two methods lend themselves to hand calculation, and the third to a computer based analysis.

12.3 Kinematic Relations

12.3.1 Force-Displacement Relations

13 Whereas in the flexibility method we sought to obtain a displacement in terms of the forces (through virtual work) for an entire structure, our starting point in the stiffness method is to develop a set of relationship for the *force in terms of the displacements for a single element*.

$$\left\{ \begin{array}{c} V_1 \\ M_1 \\ V_2 \\ M_2 \end{array} \right\} = \left[\begin{array}{cccc} - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{array} \right] \left\{ \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array} \right\} \tag{12.1}$$

14 We start from the differential equation of a beam, Fig. 12.4 in which we have all positive known displacements, we have from strength of materials

$$M = -EI\frac{d^2v}{dx^2} = M_1 - V_1x + m(x) \tag{12.2}$$

where $m(x)$ is the moment applied due to the applied load only. It is positive when counterclockwise.

15 Integrating twice

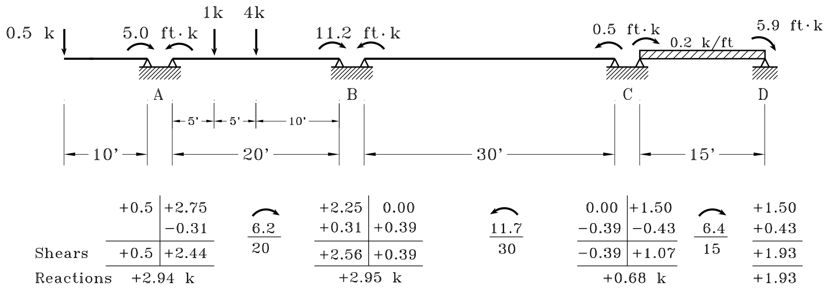
$$-EIv' = M_1x - \frac{1}{2}V_1x^2 + f(x) + C_1 \tag{12.3}$$

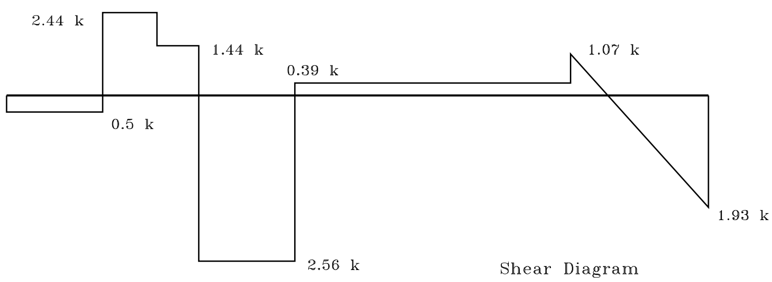
$$-EIv = \frac{1}{2}M_1x^2 - \frac{1}{6}V_1x^3 + g(x) + C_1x + C_2 \tag{12.4}$$

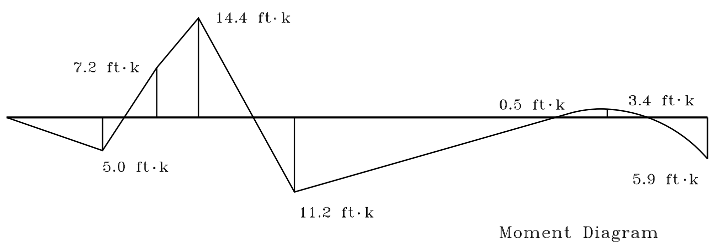
where $f(x) = \int m(x)dx$, and $g(x) = \int f(x)dx$.

16 Applying the boundary conditions at $x = 0$

$$\left. \begin{array}{rcl} v' & = & \theta_1 \\ v & = & v_1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcl} C_1 & = & -EI\theta_1 \\ C_2 & = & -EIv_1 \end{array} \right. \tag{12.5}$$



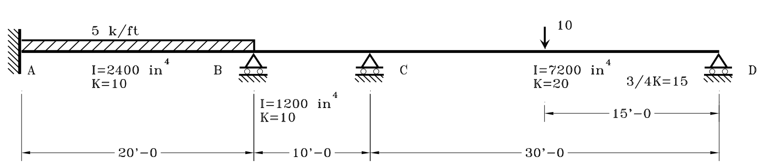




■

■ Example 12-7: Continuous Beam, Initial Settlement, (Kinney 1957)

For the following beam find the moments at A, B, and C by moment distribution. The support at C settles by 0.1 in. Use $E = 30,000$ k/in^2.

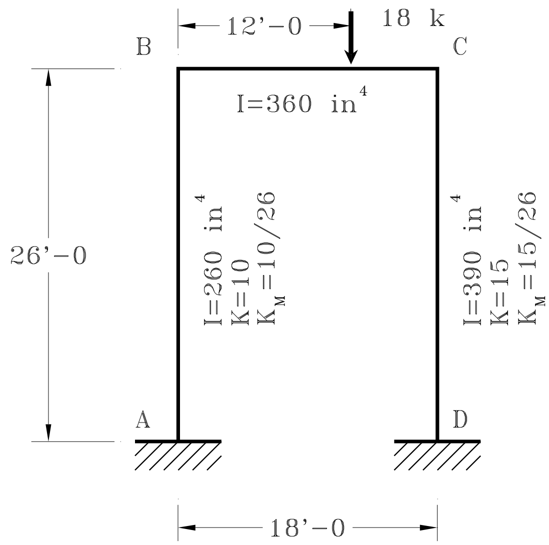


Solution:

1. Fixed-end moments: Uniform load:

$$M_{AB}^F = \frac{wL^2}{12} = \frac{(5)(20^2)}{12} = +167 \text{ k.ft} \quad (12.68\text{-a})$$

$$M_{BA}^F = -167 \text{ k.ft} \quad (12.68\text{-b})$$



Solution:

1. The first step is to perform the usual moment distribution. The reader should fully understand that this balancing operation adjusts the internal moments at the ends of the members by a series of corrections as the joints are considered to rotate, until $\Sigma M = 0$ at each joint. The reader should also realize that *during this balancing operation no translation of any joint is permitted.*
2. The fixed-end moments are

$$M^F_{BC} = \frac{(18)(12)(6^2)}{18^2} = +24 \text{ k.ft} \quad (12.71\text{-a})$$

$$M^F_{CB} = \frac{(18)(6)(12^2)}{18^2} = -48 \text{ k.ft} \quad (12.71\text{-b})$$

3. Moment distribution

Joint	*A*	*B*		*C*		*D*	Balance	CO
Member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*		
K	10	10	20	20	15	15		
DF	0	0.333	0.667	0.571	0.429	0		
FEM			+24.0	-48.0			FEM	
			+13.7	+27.4	+20.6	+10.3	C	DC; BC
	-6.3	-12.6	-25.1	-12.5			B	AB; CB
			+3.6	+7.1	+5.4	+2.7	C	BC; DC
	-0.6	-1.2	-2.4	-1.2			B	AB; CB
			+0.3	+0.7	+0.5	+0.02	C	BC; DC
		-0.1	-0.2				B	
Total	**-6.9**	**-13.9**	**+13.9**	**-26.5**	**+26.5**	**+13.2**		

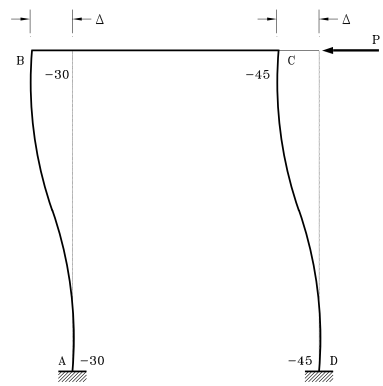
4. The *final moments listed in the table are correct only if there is no translation of any joint.* It is therefore necessary to determine whether or not, with the above moments existing, there is any tendency for side lurch of the top of the frame.
5. If the frame is divided into three free bodies, the result will be as shown below.

62 Recalling that the fixed end moment is $M^F = 6EI\frac{\Delta}{L^2} = 6EK_m\Delta$, where $K_m = \frac{I}{L^2} = \frac{K}{L}$ we can write

$$\Delta = \frac{M^F_{AB}}{6EK_m} = \frac{M^F_{DC}}{6EK_m} \tag{12.72-a}$$

$$\Rightarrow \frac{M^F_{AB}}{M^F_{DC}} = \frac{K^{AB}_m}{K^{DC}_m} = \frac{10}{15} \tag{12.72-b}$$

63 These fixed-end moments could, for example, have the values of -10 and -15 k.ft or -20 and -30, or -30 and -45, or any other combination so long as the above ratio is maintained. The proper procedure is to choose values for these fixed-end moments of approximately the same order of magnitude as the original fixed-end moments due to the real loads. This will result in the same accuracy for the results of the balance for the side-sway correction that was realized in the first balance for the real loads. Accordingly, it will be assumed that P, and the resulting Δ, are of such magnitudes as to result in the fixed-end moments shown below



8. Obviously, $\Sigma M = 0$ is not satisfied for joints B and C in this deflected frame. Therefore these joints must rotate until equilibrium is reached. The effect of this rotation is determined in the distribution below

Joint	A	B		C		D	Balance	CO
Member	AB	BA	BC	CB	CD	DC		
K	10	10	20	20	15	15		
DF	0	0.333	0.667	0.571	0.429	0		
FEM	-30.0	-30.0			-45.0	-45.0		
			+12.9	+25.8	+19.2	+9.6	C	BC; DC
	+2.8	+5.7	+11.4	+5.7			B	AB; CB
			-1.6	-3.3	-2.4	-1.2	C	BC; DC
	+0.2	+0.5	+1.1	+0.5			B	AB; CB
				-0.3	-0.2	-0.1	C	
Total	**-27.0**	**-23.8**	**+23.8**	**+28.4**	**-28.4**	**-36.7**		

9. During the rotation of joints B and C, as represented by the above distribution, *the value of* Δ *has remained constant, with* P *varying in magnitude as required to maintain* Δ.

10. It is now possible to determine the final value of P simply by adding the shears in the columns. The shear in any member, without external loads applied along its length, is obtained by adding the end moments algebraically and dividing by the length of the member. The final