Chapter 6: Equations of Motion of SDOF System for Mass as well as Base Excitation

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Introduction

In the context of Earthquake Engineering, the analysis of the dynamic response of structures is fundamental. Most structural systems can be idealized as a combination of simpler systems, of which the **Single Degree of Freedom (SDOF)** system is the most fundamental. The SDOF model captures the essence of dynamic behavior and forms the basis for understanding the response under various types of excitations — such as due to ground motion (base excitation) and external force (mass excitation).

This chapter deals in detail with deriving and analyzing the **equations of motion** for an SDOF system subjected to both **mass excitation** (external force acting directly on the mass) and **base excitation** (due to ground movement during an earthquake). Understanding these derivations is key for evaluating structural responses to seismic inputs and for designing safer and more resilient structures.

6.1 Single Degree of Freedom (SDOF) System – An Overview

- Definition: A mechanical system with only one coordinate required to describe its motion.
- Components:
    - Mass (m)
    - Spring (stiffness, k)
    - Damper (damping coefficient, c)
- Equation of motion: derived using Newton's Second Law or D'Alembert's Principle.

6.2 Free Vibration of Undamped SDOF System

- Assumptions:
  - No external force.
  - No damping.
- Governing Equation:

$$m\ddot{u}(t)+ku(t)=0$$

- Solution:

$$u(t)=A\cos(\omega_n t)+B\sin(\omega_n t)$$

where $\omega_n=\sqrt{\frac{k}{m}}$ is the natural frequency.

- Interpretation: Pure harmonic motion.

6.3 Free Vibration of Damped SDOF System

- Introduction of viscous damping:

$$m\ddot{u}(t)+c\dot{u}(t)+ku(t)=0$$

- Damping Ratio:

$$\zeta=\frac{c}{2\sqrt{mk}}$$

- Types of Damping:
  - Underdamped ($\zeta<1$)
  - Critically damped ($\zeta=1$)
  - Overdamped ($\zeta>1$)

6.4 Forced Vibration due to External Force (Mass Excitation)

6.4.1 Equation of Motion for External Force

- Applied Force: $F(t)$

- Equation:

$$m\ddot{u}(t)+c\,\dot{u}(t)+ku(t)=F(t)$$

- Methods of Solution:
  - Classical method (Duhamel's integral)
  - Laplace transform
  - Fourier Transform
  - Numerical methods (e.g., Newmark-beta)

6.4.2 Response to Harmonic Loading

- Applied force: $F(t)=F_0\sin(\omega t)$
- Steady-state solution:

$$u(t)=U\sin(\omega t-\phi)$$

  where $U=\frac{F_0}{\sqrt{¿¿¿}}$, and ϕ is the phase angle.

- Resonance condition: $\omega\approx\omega_n$

6.5 Seismic Excitation as Base Motion

During an earthquake, the ground itself moves, and hence the excitation is applied at the base of the structure rather than on the mass. This is modeled as base excitation.

6.5.1 Concept of Base Excitation

- Ground displacement: $u_g(t)$
- Relative displacement of mass with respect to the base: $u(t)$
- Absolute displacement: $u_a(t)=u(t)+u_g(t)$

6.6 Derivation of Equation of Motion for Base Excitation

6.6.1 Free Body Diagram Analysis

- Consider the structure's base moving with ground motion $u_g(t)$.
- The relative motion $u(t)$ of the mass with respect to base is governed by:

$$m\ddot{u}(t)+c\dot{u}(t)+ku(t)=-m\ddot{u}_g(t)$$

- Interpretation:
  - The RHS is a **pseudo-force** due to ground acceleration.
  - This equation shows that seismic excitation acts as an **inertial force**.

6.7 Absolute vs Relative Motion

- Absolute motion:

$$u_a(t)=u(t)+u_g(t)$$

- Engineers are often interested in **relative displacement**, as it reflects the deformation within the structure.
- However, **absolute acceleration** is important when analyzing the force demand on non-structural components (like equipment).

6.8 Solution Approaches for Base Excitation Problems

6.8.1 Duhamel's Integral

Used for linear systems with arbitrary ground motion:

$$u(t)=-\frac{1}{m}\int_0^t h(t-\tau)m\ddot{u}_g(\tau)d\tau$$

where $h(t)$ is the impulse response function.

6.8.2 Numerical Methods

- Newmark-beta method
- Wilson-theta method
- Step-by-step integration using ground motion time history data

These methods are commonly used in earthquake simulation software (e.g., ETABS, SAP2000).

6.9 Response Spectra Concept

- Peak response (displacement, velocity, acceleration) of an SDOF system for a given damping ratio subjected to a specific ground motion.
- Used to quickly estimate the maximum response without solving the full differential equation.

6.10 Comparison: Mass vs Base Excitation

Feature	Mass Excitation	Base Excitation
Applied Force	Direct on mass	Indirect via ground motion
RHS of EOM	$F(t)$	$-m\ddot{u}_g(t)$
Example	Wind load, impact	Earthquake shaking

6.11 Practical Considerations in Earthquake Engineering

- Importance of damping estimation in seismic analysis.
- Necessity of accurate ground motion records for base excitation problems.
- Simplifications in SDOF modeling for complex multi-degree-of-freedom (MDOF) structures.
- Code provisions (IS 1893) use base excitation concepts in structural design.

6.12 Idealization of Structures as SDOF Systems

In real-life applications, most structures have multiple degrees of freedom (MDOF), but for simplified analysis, especially during preliminary design or conceptual understanding, complex structures are often **idealized as SDOF systems**.

6.12.1 Criteria for Idealization

- Regularity in mass and stiffness distribution.
- Dominant first mode of vibration.
- Structures with rigid diaphragms and uniform mass distribution (e.g., low-rise buildings).

6.12.2 Lumped Mass Idealization

- Concentrate mass at floor levels.

- Connect masses with springs representing column stiffness.
- Damping is distributed based on expected energy dissipation mechanisms.

6.12.3 Translational vs Rotational SDOF

- Translational SDOF systems involve horizontal/vertical movement.
- Rotational SDOF systems model base rocking or overturning of slender structures.

6.13 Effect of Damping on Seismic Response

Damping plays a critical role in reducing the amplitude of oscillations. The **damping ratio** influences how much energy is dissipated during cyclic loading, such as seismic activity.

6.13.1 Typical Damping Values

Structure Type	Damping Ratio (ζ)
Steel Structures	2% – 5%
Reinforced Concrete	5% – 7%
Wood Structures	3% – 10%
Base-Isolated Systems	Up to 15% – 30%

6.13.2 Hysteretic Damping

- Arises from **inelastic deformation** of materials and joints.
- Modeled using **equivalent viscous damping** in linear analysis.

6.14 Role of Initial Conditions in SDOF Response

Initial displacement and velocity play a significant role in determining the system's response, especially for **underdamped** systems.

- General solution:

$$u(t) = u_h(t) + u_p(t)$$

  where $u_h(t)$ is the homogeneous (free) response due to initial conditions, and $u_p(t)$ is the particular solution due to applied forces or ground motion.

- For earthquake engineering, initial conditions may be zero, but in **aftershock** or **sequential excitation** cases, they must be considered.

6.15 Design Implications Based on SDOF Seismic Response

Understanding the SDOF system behavior under seismic excitation helps guide important design decisions:

6.15.1 Determination of Design Base Shear

Using response spectra (IS 1893):

$$V_b = A_h \cdot W$$

where A_h is the design horizontal seismic coefficient and W is the seismic weight.

6.15.2 Ductility Demand

- SDOF models help in estimating **ductility** (μ = maximum displacement/yield displacement).
- Design codes use **response reduction factors (R)** which depend on ductility and overstrength.

6.16 Influence of Ground Motion Characteristics

The equation of motion response is significantly influenced by:

6.16.1 Peak Ground Acceleration (PGA)

- Directly affects the right-hand side of the equation $-m\ddot{u}_g(t)$.
- Higher PGA results in larger inertia forces.

6.16.2 Duration of Motion

- Long-duration motions can cause **cumulative damage**, especially under repeated yielding cycles.

6.16.3 Frequency Content and Resonance

- If the predominant frequency of the ground motion matches the natural frequency of the system, **resonance** may occur, resulting in amplified response.

6.17 Limitations of SDOF Modeling

While SDOF models are useful, they have limitations:

- Cannot accurately model higher mode effects.
- Unsuitable for **irregular**, **asymmetrical**, or **tall** structures.
- Does not capture **torsional** effects.
- Requires careful selection of equivalent parameters (mass, stiffness, damping).

For accurate seismic performance evaluation, MDOF or finite element modeling (e.g., ETABS, STAAD.Pro, SAP2000) is necessary.

6.18 Advanced Concepts Linked to SDOF Systems

6.18.1 Response History Analysis

- Time history of ground motion is used as input.
- Displacement, velocity, and acceleration response of SDOF are plotted against time.

6.18.2 Incremental Dynamic Analysis (IDA)

- SDOF models are subjected to scaled versions of ground motion.
- Useful in **performance-based design**.

6.18.3 Base Isolation and Tuned Mass Dampers

- Base-isolated buildings can be modeled using modified SDOF systems.
- Tuned mass dampers (TMDs) are designed based on SDOF principles to reduce response by introducing a secondary SDOF system.

6.19 Application in Earthquake-Resistant Design

- Simplified seismic coefficient method uses SDOF assumptions.
- Push-over analysis reduces MDOF to an equivalent SDOF for performance evaluation.
- Nonlinear SDOF response helps in understanding post-yield and collapse behavior.
