Chapter 14: Parseval's Theorem

Introduction

In engineering mathematics, especially in signal processing and structural analysis, the concept of energy of a function is fundamental. **Parseval's Theorem** is a crucial result in Fourier Analysis that equates the total energy of a signal in time domain to its energy in the frequency domain. It provides a powerful bridge between the physical and spectral interpretations of functions and is highly relevant in civil engineering for analyzing periodic loads, vibrations, and stress analysis using harmonic components.

Understanding Parseval's Theorem is essential for civil engineers dealing with computational mechanics, structural dynamics, and solving partial differential equations through Fourier methods.

14.1 Mathematical Preliminaries

Before stating and proving Parseval's Theorem, we review the essentials of **Fourier series**, as the theorem directly applies to them.

Let $f(x)$ be a periodic function defined on $[-L, L]$ and integrable on this interval. Then, the **Fourier series** of $f(x)$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

where the Fourier coefficients are defined as:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1$$

14.2 Statement of Parseval's Theorem

Parseval's Theorem relates the square-integrable norm (energy) of a function over a period to the sum of the squares of its Fourier coefficients.

> **Theorem (Parseval's Identity):**
>
> If $f(x)$ is a real, periodic function with period $2L$, which is square integrable over $[-L, L]$, then:
>
> $$\frac{1}{L}\int_{-L}^{L} |f(x)|^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

This result means the total **mean square value** (or energy) of $f(x)$ over the interval is equal to the sum of the squares of its Fourier coefficients.

14.3 Derivation of Parseval's Theorem

We start with the Fourier series of $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

To compute the integral $\frac{1}{L}\int_{-L}^{L} f(x)^2 dx$, we substitute the Fourier series expansion:

$$\frac{1}{L}\int_{-L}^{L} f(x)^2 dx = \frac{1}{L}\int_{-L}^{L} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)\right]^2 dx$$

Now expanding the square and using the orthogonality of sine and cosine functions:

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = \begin{cases} L, & n = m \neq 0 \\ 0, & n \neq m \end{cases}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \begin{cases} L, & n = m \\ 0, & n \neq m \end{cases}$$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = 0 \quad \text{(for all } n, m)$$

After evaluating the integrals and simplifying, all cross terms vanish due to orthogonality, and we are left with:

$$\frac{1}{L}\int_{-L}^{L} f(x)^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

This completes the derivation.

14.4 Applications of Parseval's Theorem in Civil Engineering

Parseval's Theorem is particularly useful in the following applications:

1. Structural Vibration Analysis When a structure like a beam or a slab is subject to periodic loads, the response can be analyzed in terms of its frequency components using Fourier series. Parseval's Theorem allows engineers to compute the energy content or RMS value of vibration signals from the spectrum.

2. Signal Energy in Remote Sensing or Earthquake Data The seismic signals measured from structures or the earth's surface are periodic in nature. Parseval's Theorem enables energy calculation directly from Fourier coefficients, which helps in structural health monitoring and earthquake engineering.

3. Solving Partial Differential Equations In problems involving heat conduction (Fourier's Law) or wave equations in materials, Parseval's Theorem can be used to compute total heat or energy distributed in the system over time or space.

4. Modal Analysis in Dynamic Systems In modal analysis of multidegree-of-freedom systems (MDOF), the Fourier coefficients represent modal amplitudes. Parseval's Theorem helps in calculating the contribution of each mode to the total system energy.

14.5 Parseval's Theorem in Complex Form

When using the **complex form of Fourier series**, Parseval's Theorem takes the form:

Let:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega x}, \quad \omega = \frac{\pi}{L}$$

Then:

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx = \sum_{n=-\infty}^{\infty}|c_n|^2$$

This is especially useful in advanced engineering applications involving phasor or complex signal representation.

14.6 Conditions for Validity

Parseval's Theorem holds under the following conditions:

- $f(x)$ must be **square integrable** on $[-L, L]$, i.e.,

$$\int_{-L}^{L}|f(x)|^2dx < \infty$$

- The function should have **absolutely convergent** Fourier series.
- It is ideally used with **piecewise continuous** functions.

14.7 Worked Examples

Example 1: Compute the energy of a square wave using Parseval's Theorem Let $f(x)$ be a periodic square wave function of period 2π, defined on $(-\pi, \pi)$ as:

$$f(x) = \begin{cases} 1, & -\pi < x < 0 \\ -1, & 0 < x < \pi \end{cases}$$

This is an odd function, so all $a_n = 0$, and only sine terms appear. The Fourier coefficients are:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi}(-1)\sin(nx)\,dx = \frac{2}{n\pi}(1-(-1)^n)$$

So,

$$b_n = \begin{cases} \frac{4}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Now using Parseval's Theorem:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx = \sum_{n=1}^{\infty} b_n^2$$

Since $f(x)^2 = 1$ always,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} 1\,dx = \frac{1}{\pi}\cdot 2\pi = 2$$

Therefore:

$$\sum_{n=1}^{\infty} b_n^2 = 2 \Rightarrow \sum_{n\text{ odd}} \left(\frac{4}{n\pi}\right)^2 = 2 \Rightarrow \frac{16}{\pi^2}\sum_{n\text{ odd}} \frac{1}{n^2} = 2$$

This identity confirms Parseval's Theorem.

Example 2: Energy of a triangular waveform Let $f(x) = x$ on $(-\pi, \pi)$, extended as an odd function.

Being odd, only sine terms remain:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

Using integration by parts:

$$b_n = \frac{2}{\pi}\left[-\frac{x\cos(nx)}{n}|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)dx\right] = \frac{2}{\pi}\left(-\frac{\pi\cos(n\pi)}{n}\right) = \frac{2(-1)^{n+1}}{n}$$

Then:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n}\right)^2 = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

We know $\int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^3}{3}$, so:

$$\frac{1}{\pi}\cdot\frac{2\pi^3}{3} = \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

This classic result is verified via Parseval's identity.

14.8 Parseval's Theorem for Fourier Transforms

For **non-periodic functions**, the Fourier transform replaces the Fourier series. Parseval's Theorem still applies in this context but is stated in a different form.

Let $f(x) \in L^2(-\infty, \infty)$, and let its Fourier transform be:

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

Then Parseval's identity is:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

This is fundamental in **digital signal processing**, **vibration theory**, and **acoustic modeling**.

14.9 Parseval's Theorem in Engineering Practice

A. Structural Engineering (Modal Superposition) When structures like bridges or towers are excited by external forces (like wind, traffic), modal decomposition breaks response into harmonic modes. Parseval's Theorem helps compute:

- Total vibration energy = Sum of squared modal amplitudes

B. Finite Element Modeling In dynamic finite element models, system response $u(x, t)$ can be represented using basis functions (e.g., sine and cosine). The theorem assures energy conservation between physical and modal domains, helping in error estimation.

C. Signal Filtering For civil engineers working in **remote sensing** or **geotechnical instrumentation**, sensor data is often filtered. Parseval's Theorem helps ensure that filtering operations do not lose energy or distort signal integrity.

14.10 Key Conceptual Questions

1. What is the physical interpretation of Parseval's Theorem in structural dynamics?
2. How does Parseval's identity relate to energy conservation in a vibrating system?
3. Why do cross-terms vanish in Parseval's derivation using Fourier series?

4. How is Parseval's identity modified for complex Fourier series?
5. Extend Parseval's Theorem to functions defined on $[0, L]$ instead of $[-L, L]$. What changes?

14.11 Practice Exercises

1. For the function $f(x) = x$ on $(-\pi, \pi)$, derive its Fourier series and verify Parseval's identity.

2. Show that the function $f(x) = x^2$, $-\pi < x < \pi$, satisfies:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^4 dx = \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

where a_n, b_n are Fourier coefficients of x^2.

3. Use Parseval's Theorem to show:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

4. Prove Parseval's Theorem for a periodic function using complex exponential form of Fourier series.

5. Given the signal $f(t) = \sin(2\pi t) + \cos(4\pi t)$, calculate its energy over one period using Parseval's identity.
