Chapter 24: Vector Space

Introduction

The concept of vector spaces provides a unifying structure for various mathematical and engineering problems involving systems of linear equations, transformations, and functions. In civil engineering, vector spaces find applications in structural analysis, finite element methods, and fluid dynamics. Understanding the properties of vector spaces enables engineers to model and solve complex physical systems efficiently.

This chapter introduces the foundational ideas of vector spaces, subspaces, linear combinations, span, linear independence, basis, and dimension—essential components for higher-level problem-solving in engineering.

24.1 Definition of Vector Space

A **vector space** (also called a linear space) over a field **F** (usually the field of real numbers ℝ) is a set **V** equipped with two operations:

- **Vector addition**: $+ : V \times V \to V$
- **Scalar multiplication**: $\cdot : F \times V \to V$

such that the following **axioms** hold for all $u, v, w \in V$ and all scalars $a, b \in F$:

Axioms of Vector Space:

1. **Closure under addition**: $u+v \in V$
2. **Commutativity of addition**: $u+v=v+u$
3. **Associativity of addition**: $(u+v)+w=u+(v+w)$
4. **Existence of additive identity**: There exists a vector $0 \in V$ such that $v+0=v$
5. **Existence of additive inverse**: For each $v \in V$, there exists $-v \in V$ such that $v+(-v)=0$
6. **Closure under scalar multiplication**: $a \cdot v \in V$
7. **Distributivity over vector addition**: $a \cdot (u+v)=a \cdot u+a \cdot v$
8. **Distributivity over scalar addition**: $(a+b) \cdot v=a \cdot v+b \cdot v$
9. **Associativity of scalar multiplication**: $a \cdot (b \cdot v)=(ab) \cdot v$

10. **Identity scalar multiplication**: $1 \cdot v = v$, where 1 is the multiplicative identity in **F**.

24.2 Examples of Vector Spaces

1. $\mathbb{R}^n$ **(n-dimensional real space)**

Each element is an n-tuple: $(x_1, x_2, \ldots, x_n)$ with real components.

2. **Set of all real-valued functions**

Let **V** be the set of all real-valued functions $f : \mathbb{R} \rightarrow \mathbb{R}$. Then **V** is a vector space with function addition and scalar multiplication.

3. **Set of m×n real matrices**

The set $M_{m \times n}(R)$ of all m×n matrices forms a vector space under matrix addition and scalar multiplication.

4. **Set of polynomials of degree ≤ n**

The set of all polynomials with real coefficients and degree ≤ n is a vector space.

24.3 Subspace

A **subspace** of a vector space **V** is a non-empty subset **W** of **V** that is itself a vector space under the same operations.

Conditions for Subspace:

Let $W \subseteq V$. Then W is a subspace if:

1. $0 \in W$
2. $u, v \in W \Rightarrow u + v \in W$
3. $a \in R, v \in W \Rightarrow a \cdot v \in W$

Example:

Let $V = R^3$ and $W = \{(x, y, z) \in R^3 : x + y + z = 0\}$. Then W is a subspace of R^3.

24.4 Linear Combination and Span

- A **linear combination** of vectors $v_1, v_2, \ldots, v_k \in V$ is an expression of the form

  $$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$$

  where $a_i \in R$.

- The **span** of vectors $\{v_1, v_2, \ldots, v_k\}$, denoted by

  $$\text{span}\{v_1, \ldots, v_k\}$$

  is the set of all linear combinations of $v_1, \ldots, v_k$.

- Span is always a subspace of **V**.

24.5 Linear Independence

A set of vectors $\{v_1, v_2, \ldots, v_k\} \subseteq V$ is said to be **linearly independent** if

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0 \Rightarrow a_1 = a_2 = \cdots = a_k = 0$$

If there exist scalars not all zero satisfying the above equation, then the vectors are **linearly dependent**.

Example:

Vectors $(1,2),(2,4) \in R^2$ are linearly dependent, since $2(1,2) - (2,4) = (0,0)$

24.6 Basis

A **basis** of a vector space **V** is a set of linearly independent vectors that spans **V**.

- If $B = \{v_1, \ldots, v_n\}$ is a basis of **V**, then every element $v \in V$ can be uniquely written as a linear combination of the vectors in B.

Example:

The standard basis of R^3 is $\{(1,0,0),(0,1,0),(0,0,1)\}$

24.7 Dimension

The **dimension** of a vector space **V** is the number of vectors in any basis of **V**.

- If V has a finite basis with n vectors, then $dim(V)=n$
- If no finite basis exists, V is called **infinite-dimensional**

Examples:

- $dim(R^n)=n$
- The space of all polynomials of degree ≤ 2 has dimension 3.

24.8 Coordinates of a Vector

Given a basis $B=\{v_1, v_2, \ldots, v_n\}$ of a vector space **V**, any vector $v \in V$ can be written uniquely as:

$$v=a_1 v_1+a_2 v_2+\cdots+a_n v_n$$

The tuple $(a_1, a_2, \ldots, a_n)$ is called the **coordinate vector** of v with respect to the basis **B**.

24.9 Row Space, Column Space, and Null Space

Let A be an $m \times n$ matrix.

- The **row space** of A: span of the row vectors.
- The **column space** of A: span of the column vectors.
- The **null space** of A: set of all solutions $x \in R^n$ to $Ax=0$

Each of these is a subspace of a suitable vector space.

24.10 Rank and Nullity

- The **rank** of a matrix A: dimension of the column space.
- The **nullity** of A: dimension of the null space.

Rank-Nullity Theorem:

$$\text{Rank}(A)+\text{Nullity}(A)=n$$

Where A is an $m \times n$ matrix.

24.11 Applications in Civil Engineering

Structural Analysis:

Vector spaces model displacement vectors, forces, and deformations in structures like beams and frames.

Finite Element Methods (FEM):

The basis functions used in FEM form a vector space. Understanding basis and dimension helps in choosing appropriate shape functions.

Optimization and Linear Systems:

Design optimization problems and systems of equations derived from physical laws are best handled using vector space theory.

24.12 Vector Space Isomorphism

Two vector spaces V and W over the same field are **isomorphic** if there exists a bijective linear map (isomorphism) $T : V \to W$ such that:

$$T(av + bw) = aT(v) + bT(w)$$

for all $v, w \in V$ and $a, b \in R$.

Significance:

- Isomorphic vector spaces are **structurally identical**.
- If $\dim V = \dim W = n$, then $V \cong R^n$.

24.13 Direct Sum of Subspaces

Let V be a vector space, and let U and W be subspaces of V.

We say:

$$V = U \oplus W$$

if:

- Every element $v \in V$ can be uniquely written as $v = u + w$, where $u \in U, w \in W$
- $U \cap W = \{0\}$

This helps break down complex vector spaces into simpler components.

24.14 Quotient Vector Spaces

Let V be a vector space and $W \subseteq V$ a subspace.

The **quotient space** V/W is the set of equivalence classes:

$$V/W = \{v + W : v \in V\}$$

Each element of V/W is a **coset** of the form $v + W$. The operations are:

- $(v+W)+(u+W)=(v+u)+W$
- $a(v+W)=(av)+W$

Quotient spaces are critical in differential equations, numerical analysis, and finite element error estimates.

24.15 Dual Space

Given a vector space V, the **dual space** $V^{¿}$ is the set of all **linear functionals** from V to R:

$$V^{¿} = \{f : V \to R \mid f \text{ is linear}\}$$

If V is finite-dimensional with basis $\{v_1, \ldots, v_n\}$, then $V^{¿}$ also has dimension n.

In engineering, dual spaces are used in **stress-strain analysis**, where stresses are linear functionals acting on displacement fields.

24.16 Worked Examples

Example 1: Determining Subspace

Let $W = \{(x, y, z) \in R^3 : x + 2y + 3z = 0\}$. Show that W is a subspace of R^3.

Solution:

1. **Zero vector**: $(0,0,0) \in W$ since $0+2(0)+3(0)=0$
2. **Closed under addition**:

$$(x_1+x_2)+2(y_1+y_2)+3(z_1+z_2)=(x_1+2y_1+3z_1)+(x_2+2y_2+3z_2)=0+0=0$$

3. **Closed under scalar multiplication**:

$$a(x+2y+3z)=a\cdot 0=0$$

Hence, W is a subspace.

Example 2: Basis and Dimension

Find a basis for $W=\{(x,y,z)\in R^3 : x+y+z=0\}$ and its dimension.

Solution:

Let $x=-y-z$. Then any vector in W is of the form:

$$(-y-z,y,z)=y(-1,1,0)+z(-1,0,1)$$

So the basis is:

$$\{(-1,1,0),(-1,0,1)\}$$

Dimension = 2.

24.17 Orthogonality in Vector Spaces

Two vectors $u,v\in R^n$ are **orthogonal** if their dot product is zero:

$$u\cdot v=0$$

A set of vectors is **orthogonal** if every pair in the set is orthogonal. If they are also unit vectors, the set is **orthonormal**.

Orthonormal Basis:

A basis $\{v_1,v_2,\ldots,v_n\}$ is orthonormal if:

$$v_i\cdot v_j=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$$

Useful in numerical methods (e.g., Gram-Schmidt process) and matrix decompositions (e.g., QR factorization).

24.18 Gram-Schmidt Orthogonalization

Given a set of linearly independent vectors $\{v_1, v_2, \dots, v_n\}$, this process constructs an **orthonormal basis** $\{u_1, u_2, \dots, u_n\}$ such that:

$$u_1 = \frac{v_1}{\| v_1 \|}, u_2 = \frac{v_2 - \text{proj}_{u_1}(v_2)}{\| v_2 - \text{proj}_{u_1}(v_2) \|}, \dots$$

where:

$$\text{proj}_u(v) = \frac{v \cdot u}{u \cdot u} u$$

24.19 Visual Insights

To help visualize vector spaces:

- $\mathbb{R}^2$: A plane with vectors as arrows from the origin.
- **Subspaces**: Lines through the origin or planes in higher dimensions.
- **Basis**: Minimum set of independent vectors needed to describe all vectors.
- **Null Space**: The set of vectors mapped to zero — visualized as a flat region.
- **Column Space**: Span of column vectors — represents the range of transformation.

(Diagrams should accompany these explanations in your e-book for better comprehension.)

24.20 MATLAB/Python Implementation (Optional Section)

For engineering students using computational tools:

Finding Rank and Nullity in Python:

```
import numpy as np
A = np.array([[1, 2, 3], [2, 4, 6], [1, 0, 1]])
rank = np.linalg.matrix_rank(A)
```

```
null_space_dim = A.shape[1] - rank
print("Rank:", rank)
print("Nullity:", null_space_dim)
```

This section can help students apply vector space concepts to real computations in structural design, simulations, or data modeling.
