Chapter 13: Normal Modes of Vibration

Introduction

In earthquake engineering and structural dynamics, understanding how structures respond to vibrational forces is crucial. One of the most fundamental concepts in this area is **normal modes of vibration**. These are the natural ways in which a system tends to oscillate in the absence of external forces or damping. Each normal mode is associated with a specific frequency—called a **natural frequency**—and a unique deformation shape—called a **mode shape**.

Structures like buildings and bridges, when subjected to dynamic loads such as seismic waves, exhibit complex motion. However, this complex motion can be broken down into a combination of simpler vibrational patterns—normal modes. This chapter delves deep into the theoretical basis, mathematical formulation, physical significance, and practical applications of normal modes in multi-degree-of-freedom (MDOF) systems, which are common in real-world structures.

13.1 Multi-Degree-of-Freedom (MDOF) Systems

- Definition and examples of MDOF systems
- Mathematical modeling of MDOF structures (e.g., multi-storey frames)
- Equations of motion in matrix form:

$$[M]\{\ddot{X}\} + [K]\{X\} = \{F(t)\}$$

- where:
  - $[M]$ = mass matrix
  - $[K]$ = stiffness matrix
  - $\{X\}$ = displacement vector
  - $\{F(t)\}$ = external force vector
- Coupled differential equations and challenges in solving them directly

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13.2 Concept of Mode Shapes and Natural Frequencies

- Definition of **natural frequency** and **mode shape**
- Physical interpretation of normal modes: oscillation patterns independent of each other
- Mode shapes as eigenvectors and natural frequencies as eigenvalues

- Orthogonality of mode shapes with respect to mass and stiffness matrices

$$[\Phi]^T[M][\Phi] = [I], \quad [\Phi]^T[K][\Phi] = [\Lambda]$$

- where:
  - $[\Phi]$ = modal matrix (mode shapes as columns)
  - $[\Lambda]$ = diagonal matrix of squared natural frequencies
  - $[I]$ = identity matrix

13.3 Free Vibration Analysis of MDOF Systems

- Assumption of no damping and no external force
- Solution using harmonic motion:

$$\{X(t)\} = \{\phi\} \sin(\omega t)$$

- Substituting into equations of motion leads to:

$$([K] - \omega^2[M])\{\phi\} = 0$$

- This forms an eigenvalue problem for ω^2 and $\{\phi\}$
- Procedure:
  a. Formulate [M] and [K]
  b. Solve characteristic equation $\det([K] - \omega^2[M]) = 0$
  c. Obtain natural frequencies ω_n and corresponding mode shapes $\{\phi_n\}$

13.4 Properties of Normal Modes

- **Orthogonality Property**: Mode shapes are orthogonal w.r.t. both [M] and [K]

$$\{\phi_i\}^T[M]\{\phi_j\} = 0, \quad i \neq j$$

$$\{\phi_i\}^T[K]\{\phi_j\} = 0, \quad i \neq j$$

- **Normalization**: Mode shapes can be scaled to satisfy unit modal mass:

$$\{\phi_n\}^T[M]\{\phi_n\} = 1$$

- **Modal Participation Factors**: Indicate how much a mode contributes to the total response
- **Completeness**: Any dynamic response of a linear system can be expressed as a linear combination of its normal modes

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13.5 Modal Analysis Technique

- **Objective**: To decouple coupled differential equations using modal transformation
- Define modal coordinates:

$$\{X\} = [\Phi]\{q\}$$

- where $\{q\}$ = modal coordinate vector
- Substituting into the original equation:

$$[M][\Phi]\{\ddot{q}\} + [K][\Phi]\{q\} = \{F(t)\}$$

- Using orthogonality properties:

$$\{\ddot{q}\} + [\Lambda]\{q\} = [\Phi]^T\{F(t)\}$$

- Result: A set of uncoupled single-degree-of-freedom (SDOF) equations
- Each equation can be solved independently:

$$\ddot{q}_n + \omega_n^2 q_n = F_n(t)$$

- The total response:

$$\{X(t)\} = \sum_{n=1}^{N} \{\phi_n\} q_n(t)$$

13.6 Application in Earthquake Engineering

- **Response Spectrum Analysis**: Estimation of maximum response using modal properties
- **Seismic Design**: Mode shapes guide where reinforcements and base isolators are most needed
- **Modal Combination Methods**:

  - SRSS (Square Root of the Sum of the Squares)
  - CQC (Complete Quadratic Combination)
  - Useful when modes are closely spaced

- **Simplification in Multi-Storey Buildings**:
  - First few modes dominate the response
  - Higher modes have negligible influence in low-rise buildings

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13.7 Computational Aspects

- Use of matrix algebra and numerical eigenvalue solvers
- Application of software tools (e.g., MATLAB, SAP2000, ETABS)
- Need for accurate [M] and [K] matrices
- Sensitivity of natural frequencies to mass and stiffness distributions

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13.8 Examples and Case Studies

- **Two-degree-of-freedom system**: Detailed step-by-step computation of natural frequencies and mode shapes
- **Three-storey shear building**: Modal analysis, participation factors, and reconstruction of time response
- **Comparison with real earthquake data**: Validation of modal analysis predictions

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13.9 Effect of Damping on Mode Shapes

- **Damped Systems vs. Undamped Systems**:
  - In undamped systems, mode shapes are purely real and orthogonal.
  - In damped systems, damping may couple the modes, especially if damping is non-proportional.

- **Types of Damping**:
  - **Classical (Proportional) Damping**:

$$[C] = \alpha[M] + \beta[K]$$

- Leads to uncoupled modal equations.

- **Non-Classical (Non-Proportional) Damping**: Leads to complex modes and complex eigenvalues.

- **Modal Damping Ratios**:

– Defined for each mode:

$$\zeta_n = \frac{c_n}{2\sqrt{k_n m_n}}$$

- $\zeta_n < 1$: Underdamped
- $\zeta_n = 1$: Critically damped
- $\zeta_n > 1$: Overdamped

13.10 Mode Truncation and Modal Superposition

- **Mode Truncation**:
  - In practice, only a few dominant modes are sufficient to approximate structural response.
  - Higher modes are neglected if their participation is negligible.
  - Error introduced by truncation is evaluated using **modal mass participation** or **energy contribution**.
- **Modal Superposition**:
  - Total system response obtained by summing individual modal responses.

$$\{X(t)\} \approx \sum_{n=1}^{r} \{\phi_n\} q_n(t) \quad \text{where } r \ll N$$

- **Criteria for Acceptable Truncation**:
  - At least 90–95% of total mass participation should be captured.
  - Dominant modes in the direction of excitation are prioritized.

13.11 Coupled Modes in Asymmetric and Torsional Systems

- Structures with unsymmetrical mass or stiffness exhibit **coupled translational and rotational modes**.
- **Torsional Modes**:
  - Significant in irregular buildings and structures with eccentric mass/stiffness.
  - Dangerous due to stress concentration and damage in corners.
- Example:
  - Plan-asymmetric buildings showing **torsionally coupled** mode shapes.

- Importance in seismic design due to uneven drift and base shear.

13.12 Experimental Determination of Mode Shapes

- **Modal Testing Methods**:
  - **Impact Hammer Testing**
  - **Shaker Testing**
  - **Ambient Vibration Testing**
- **Measurement Tools**:
  - Accelerometers, laser vibrometers, and strain gauges.
- **Frequency Response Function (FRF)**:
  - Used to identify natural frequencies and mode shapes.
- **Operational Modal Analysis (OMA)**:
  - Conducted under ambient (natural) excitation like wind or micro-tremors.
  - Useful for existing buildings and bridges without artificial excitation.

13.13 Importance of Mode Shapes in Seismic Design Codes

- **Building Code Requirements**:
  - IS 1893 (Part 1): Modal analysis is mandatory for:
    * Buildings > 40 m height in seismic zones II–V
    * Buildings with irregular configuration
- **Design Implications**:
  - Placement of shear walls, braces, and dampers based on mode shape patterns.
  - Floor accelerations and inter-storey drift predictions rely on accurate mode shapes.
- **Dynamic Load Distribution**:
  - Base shear distribution across storeys depends on modal participation.
