Chapter 2: Homogeneous Linear Equations of Second Order

Introduction

In Civil Engineering, the behavior of structural components such as beams, bridges, and buildings often involves second-order differential equations. These equations arise in the analysis of mechanical vibrations, heat conduction, fluid flow, and elasticity. Specifically, *homogeneous linear second-order differential equations* are central to mathematical modeling in these systems. This chapter explores the theory and solution methods of such equations, providing a foundation for understanding real-world engineering phenomena.

2.1 Definition

A **second-order linear homogeneous differential equation** has the general form:

$$a(x)\frac{d^2y}{dx^2}+b(x)\frac{dy}{dx}+c(x)y=0$$

Where:

- $y=y(x)$ is the unknown function of the independent variable x
- $a(x), b(x), c(x)$ are given functions of x
- $a(x)\neq 0$

If $a(x), b(x), c(x)$ are constants, the equation is said to have **constant coefficients**.

2.2 Homogeneous Linear Equations with Constant Coefficients

The most common and solvable form in engineering applications is:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

Dividing through by a (assuming $a \neq 0$):

$$\frac{d^2 y}{d x^2} + p \frac{d y}{d x} + q y = 0$$

Where:

- $p = \frac{b}{a}$,
- $q = \frac{c}{a}$

2.3 Auxiliary Equation and General Solution

To solve:

$$\frac{d^2 y}{d x^2} + p \frac{d y}{d x} + q y = 0$$

We assume a solution of the form:

$$y = e^{m x}$$

Substituting into the differential equation:

$$m^2 e^{m x} + p m e^{m x} + q e^{m x} = 0$$

$$e^{m x} \left(m^2 + p m + q \right) = 0$$

Since $e^{m x} \neq 0$, the **auxiliary equation (characteristic equation)** is:

$$m^2 + p m + q = 0$$

The nature of the roots of the auxiliary equation determines the form of the general solution.

2.4 Case I: Real and Distinct Roots

If the auxiliary equation has two distinct real roots m_1 and m_2, then:

$$y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

Where C_1, C_2 are arbitrary constants determined by initial/boundary conditions.

2.5 Case II: Real and Repeated Roots

If the roots are real and equal, say $m_1 = m_2 = m$, then:

$$y(x) = (C_1 + C_2 x) e^{mx}$$

This accounts for the multiplicity of the solution space.

2.6 Case III: Complex Roots

If the roots are complex conjugates:

$$m = \alpha \pm i\beta$$

Then the general solution becomes:

$$y(x) = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

This form is particularly useful in modeling damped vibrations or oscillatory motion.

2.7 Examples

Example 1: Distinct Real Roots

Solve:

$$\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Solution: Auxiliary equation: $m^2 - 5m + 6 = 0$ Roots: $m = 2, 3$ General solution:

$$y(x) = C_1 e^{2x} + C_2 e^{3x}$$

Example 2: Repeated Roots

Solve:

$$\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Solution: Auxiliary equation: $m^2 - 4m + 4 = 0 \Rightarrow$ ¿ Roots: $m = 2$ (repeated) Solution:

$$y(x) = (C_1 + C_2 x) e^{2x}$$

Example 3: Complex Roots

Solve:

$$\frac{d^2 y}{d x^2} + 4y = 0$$

Solution: Auxiliary equation: $m^2 + 4 = 0 \Rightarrow m = \pm 2i$ General solution:

$$y(x) = C_1 \cos 2x + C_2 \sin 2x$$

2.8 Application in Civil Engineering

1. Vibrations of Beams

The deflection of a simply supported beam subject to dynamic loading can be modeled as:

$$EI \frac{d^4 y}{d x^4} + m \frac{d^2 y}{d t^2} = 0$$

Under certain simplifications (e.g., uniform beam, time-independent loading), this reduces to a second-order homogeneous linear equation.

2. Thermal Analysis

Temperature distribution in rods or walls can lead to equations of the form:

$$\frac{d^2 T}{d x^2} - \lambda T = 0$$

Which is a second-order homogeneous equation.

3. Structural Mechanics

In analyzing column stability (Euler's buckling), we derive:

$$\frac{d^2 y}{d x^2} + \frac{P}{EI} y = 0$$

Which again is a linear homogeneous second-order differential equation.

2.9 Exercises

1. Solve $\frac{d^2y}{dx^2}+7\frac{dy}{dx}+12y=0$
2. Solve $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=0$
3. Solve $\frac{d^2y}{dx^2}-10y=0$
4. A structure's vibration is modeled as $\frac{d^2y}{dx^2}+16y=0$. Find its solution.
5. Prove that the general solution of a second-order linear homogeneous ODE always contains two arbitrary constants.

Certainly! Here's **additional content** for your e-book right after **Section 2.9 Exercises**, continuing the depth and keeping it engaging for B.Tech Civil Engineering students. This content includes solved exercises, graphical interpretation, numerical methods overview, and real-world engineering problems.

2.10 Solved Exercises

Exercise 1

Problem: Solve $\frac{d^2y}{dx^2}+7\frac{dy}{dx}+12y=0$

Solution: Auxiliary equation:

$$m^2+7m+12=0$$

Solving:

$$m=\frac{-7\pm\sqrt{49-48}}{2}=\frac{-7\pm1}{2}\Rightarrow m_1=-3, m_2=-4$$

Hence, the general solution is:

$$y(x)=C_1e^{-3x}+C_2e^{-4x}$$

Exercise 2

Problem: Solve $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=0$

Solution: Auxiliary equation:

$$m^2+6m+9=0 \Rightarrow ¿$$

General solution:

$$y(x)=(C_1+C_2x)e^{-3x}$$

Exercise 3

Problem: Solve $\frac{d^2y}{dx^2}-10y=0$

Solution: Auxiliary equation:

$$m^2-10=0 \Rightarrow m=\pm\sqrt{10}$$

General solution:

$$y(x)=C_1e^{\sqrt{10}x}+C_2e^{-\sqrt{10}x}$$

2.11 Graphical Interpretation of Solutions

Understanding the **shape and behavior** of the solution curves is essential for engineering intuition:

Type of Roots	Solution Form	Graphical Behavior
Real and Distinct	$y=C_1e^{m_1x}+C_2e^{m_2x}$	Exponential growth/decay (non-oscillatory)
Real and Repeated	$y=(C_1+C_2x)e^{mx}$	Exponential decay with linear modification
Complex Roots	$y=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta$	Oscillatory (sine-wave like), possibly damped

☞ Engineers use these plots to interpret phenomena like *damped oscillations*, *stability*, and *resonance* in systems.

2.12 Numerical Methods Overview

In real-life engineering scenarios, it is not always possible to find exact solutions. When dealing with:

- Complicated boundary conditions
- Nonlinear extensions
- Irregular material properties

... we resort to **numerical techniques**.

Euler's Method (Basic Idea)

Used for solving initial value problems:

Given:

$$\frac{d^2 y}{d x^2}=f(x, y, y'), y(x_0)=y_0, y'(x_0)=y_{0'}$$

We convert the second-order ODE into a system of first-order ODEs and apply step-by-step approximations.

Though basic, Euler's method is foundational to understanding advanced techniques like:

- **Runge-Kutta Methods**
- **Finite Difference Methods (FDM)**
- **Finite Element Methods (FEM)** – very important in civil engineering!

2.13 Real-World Civil Engineering Applications

1. Beam Deflection under Load

Using the *Euler-Bernoulli Beam Theory*:

$$E I \frac{d^4 y}{d x^4}=q(x)$$

In simplified cases with symmetric loadings, this reduces to a second-order equation involving slope and curvature:

$$\frac{d^2 y}{d x^2} + k y = 0$$

2. Column Buckling (Euler's Buckling Theory)

For a column under axial load P:

$$\frac{d^2 y}{d x^2} + \frac{P}{E I} y = 0$$

This helps determine the **critical load** beyond which the structure becomes unstable.

2.14 Key Engineering Insights

- **Boundary conditions** dictate physical behavior — for example, fixed ends or free ends of a beam.
- **Natural frequencies** arise from complex roots — important in earthquake analysis.
- **Solution behavior** changes drastically based on sign and nature of coefficients p and q.

2.15 Summary Points

- A second-order linear homogeneous equation has the general form:

$$\frac{d^2 y}{d x^2} + p \frac{d y}{d x} + q y = 0$$

- Solution behavior depends on the **discriminant** $D = p^2 - 4q$
- Real-life engineering models often lead to such equations
- Numerical methods are essential when exact solutions are not feasible
- Understanding solution graphs aids in practical design and safety analysis

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