Hydraulic Engineering
Prof. Mohammad Saud Afzal
Department of Civil Engineering
Indian Institute of Technology-Kharagpur

Lecture # 60
Introduction to wave mechanics (Contd.)

Welcome back students. In the last lecture we studied the boundary conditions in general and we saw what the kinematic boundary condition means we derived an equation of the general surface and the kinetic boundary condition related to that surface. In this lecture, we proceed forward with the bottom boundary conditions also called as, BBC.

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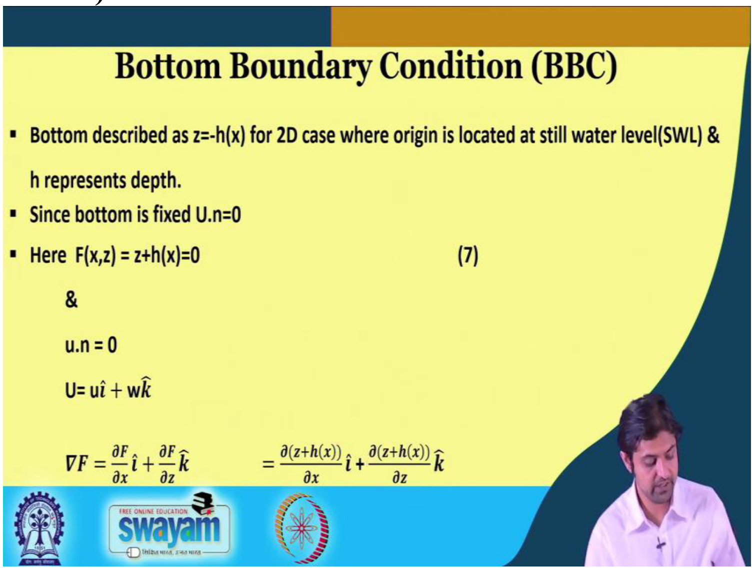

So, let us say the bottom is described as z = - h of x. So, if there is you know this is the riverbed or the seabed and this can be z here. So, if this is x and this is z. So, this depth z = - h f x because, we are considering the 0 at the free surface if we consider 0 at the free surface. So, here origin is located at still water level that is the surface of the water. So, again trying this is the origin let us say this is this.

Since from the boundary condition we see the bottom is fixed for you dot n is going to be 0, which we have seen in the previous lecture. So, we can ride the surface equation as so the equation was z = - h of x. So, we can write 0 + h of x 1 0 and we call it as f function of x and z all. And u dot n = 0 if we assume u is a combination of small ui + w k so, this is the general you that we can assume in 2 dimension x and z.

So, we have we are going to apply what we have learned in the previous lecture. So, delta F is going to be delta F delta x + delta F delta z this is an i direction this is in k direction. And instead of this f we put z + x h. So, what we get is delta Del x of z + h x here icap + Del z + h x at k cap.

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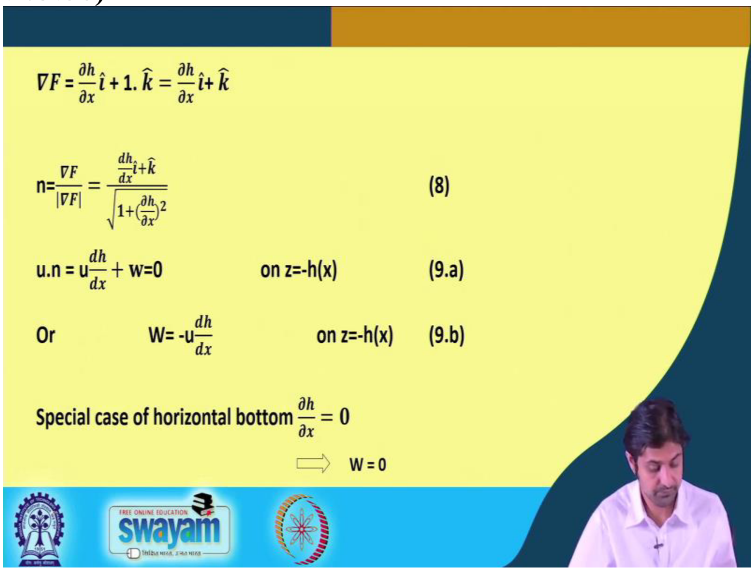

So, delta f will be del icap + k cap. This will be delta F. So n will be delta F divided by modulus of delta x so we will get dh dx i cap + k divided by 1 + dh dx whole squared simple mathematics. So and u dot n u is substitute u i cap + w k cap and n we already know this was n. So if we do dot product of u dot n what do what we get we get this multiplied by this multiplied by this. So, we get u dh dx + w is equal because here whatever is there 1 + the delta h delta x whole squared = 0.

Therefore, if it goes this side we get u dh dx + w = 0 this is u velocity and this is w velocity, u velocity means x direction w means z direction so, we get u dot n = u dh dx + w = 0 on the surface or we can simply if we use this equation, we write w = - u dh dx on the bottom boundary. Now, earlier we had assumed that the bottom I mean z will be - h of x, all. But if you assume a case of horizontal bottom in horizontal bottom, we know that the height will not vary as a function of x.

So, dh dx is going to be 0 for horizontal bottom, which we have written here. Therefore this means w = 0. So, we have now proved that the boundary w = 0 using the bottom boundary condition and we have derived everything from the basics of the boundary condition.

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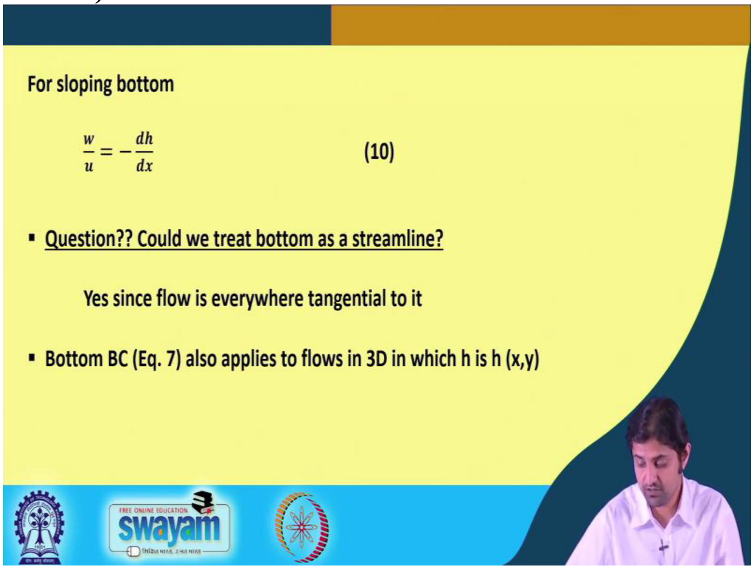

Now, if we have a sloping bottom, we can simply in case of a sloping bottom like this, we can write w by u = - dh dx very simple. The question to you is, could we treat this bottom as a streamline or not? Yes, since the flow is everywhere tangential to it. So this bottom can be treated as a streamline because w = 0 and therefore, the flow is everywhere tangential to it to the bottom boundary condition, which was equation number seven also applies to flows in 3D in which h is h of x, y. You remember we started derivation with z = - h of x.

But let us say z is not only a function of h of but it is also a function of x and y. Then also that bottom boundary condition equation seven will apply. Just it will have 1 more term Indeed, differential equation that we have.

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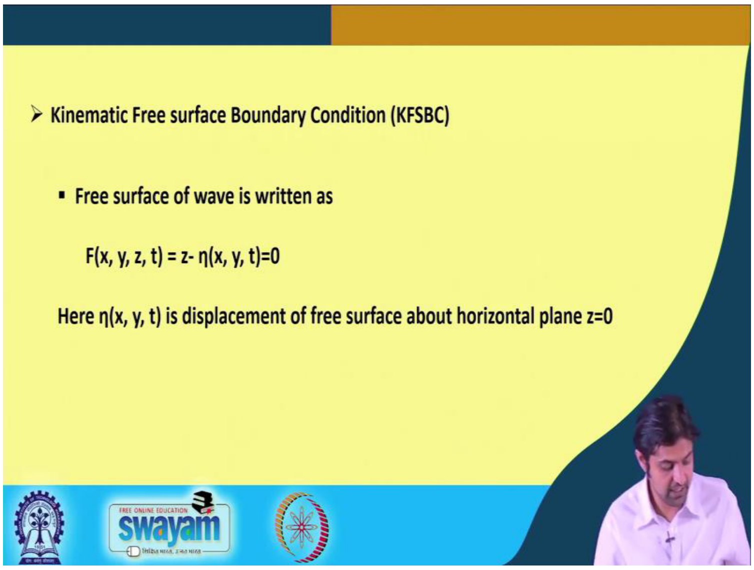

So, another term is kinematic free surface see, first we studied the boundary condition, we studied kinetic boundary condition in detail. So, the first sub part of that was we saw the bottom boundary condition where the boundary was fixed, where we utilize u dot n = 0. Now, there is something called dynamic free surface boundary condition, free surface means, free surface of water is written as. So, you the surface is free it can distort if you recall from our open channel flow lectures.

So, if the surface is something like this, you know then the free surface and this is it as a function of x y and t, then we can write F of x y z, t = z – eta x y t. So, here eta x y t is the displacement of free surface about horizontal planes z = 0 this is z = 0.

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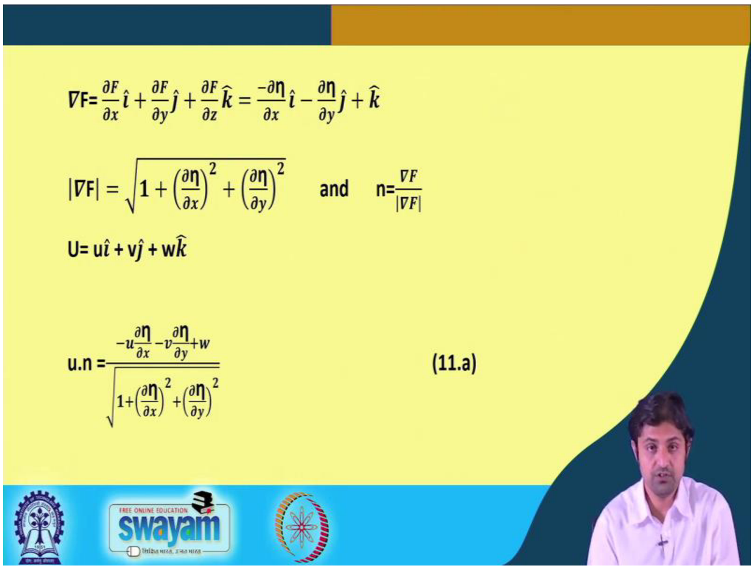

So delta F, so, we are going to apply the same conditions the kinematic boundary conditions, so, we start calculating delta F because we already have an equation F here. So, first step is always calculating delta F. So, delta F is going to be delta F delta x i cap + delta F delta y Jacob + delta F delta z k cap, so, if we put F as here z - eta x y t, then we get – del eta del x – del eta del y + k cap, so, - del eta del x i cap – del eta del y j cap + k cap. So, because then it will be 1.k cap because we had the term z here, you see alone term z.

And delta F is going to be if you take the model as it will be 1 + delta x delta said modulus is taken from here, you know this term delta x delta eta by delta x whole squared + delta eta by delta y whole squared and the unit normal vector will be delta F by modulus of delta. If we assume these velocities 3 dimensional ui vj and wk then u dot n is going to be - u del eta del x - v del eta del y + w. So, u is ui + vj + wk and eta is - del n cap is del eta del x i cap – del eta del y j cap + k cap.

If you do u dot n 1 = 0. So, if you see i would i n i will become - u del eta del x + - of v del eta del y + w or divided by of course, the denominator, 1 + del eta del x whole squared del eta del y whole squared. So, u dot n is going to be - u del eta del x - v del eta del y + w divided by the denominator same as this 1 here. So, this is what we get u dot n.

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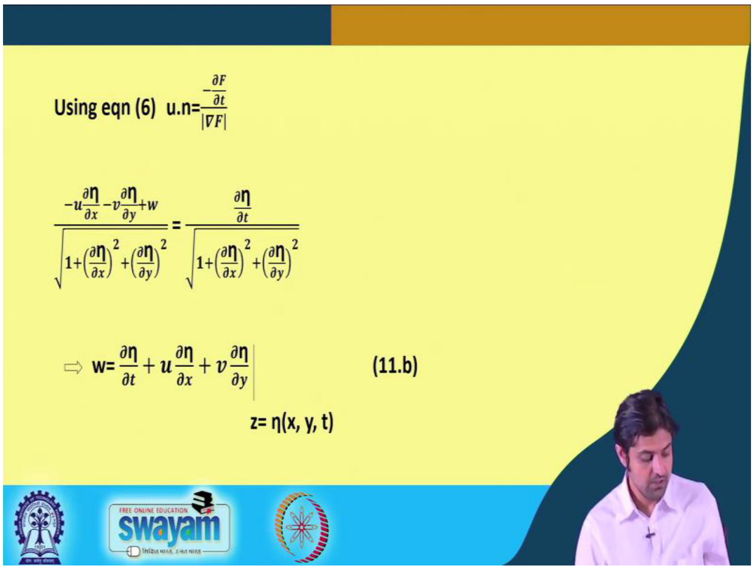

Now, if you use equation 6 was this. So, this is u dot n and if we do del F del t it will be del eta del t divided by modulus of F this and this will get cancelled out. So, we will get w = del eta del t + u del eta del x + v del eta del y at z = eta of y x t that is at the free surface this is the equation of the free surface this is an important equation. So, this is the dynamic free surface boundary condition.

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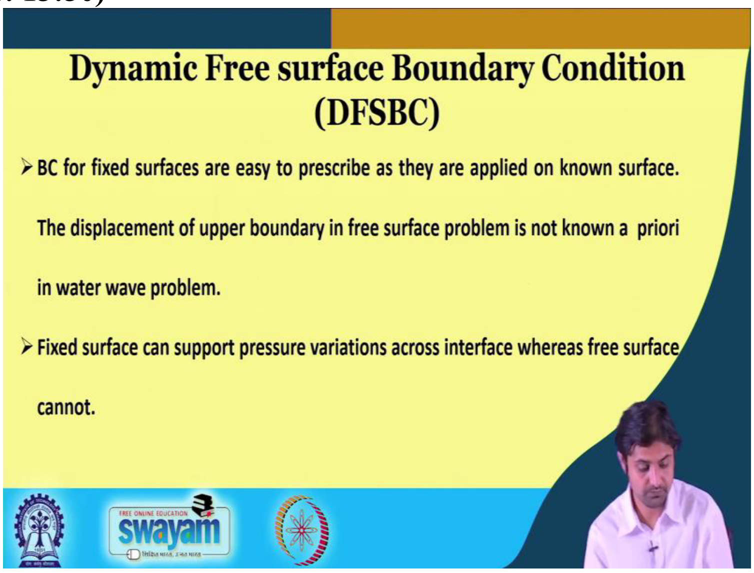

So there is another boundary condition called dynamic free surface boundary condition. So boundary condition for fixed surfaces are easy to prescribe as they are applied to the known surface. However the displacement of the upper boundary in free surface problem is not known, we do not know from before how much that the free surface will go above because I mean the bottom is fine because it is fixed, but the displacement of the free surface is not known from before.

So, that is one of the problem in the wave problem in waterway problem. Second thing is that the fixed surface are able to support pressure variations across the interface whereas, free surface cannot support the pressure variations it will distort.

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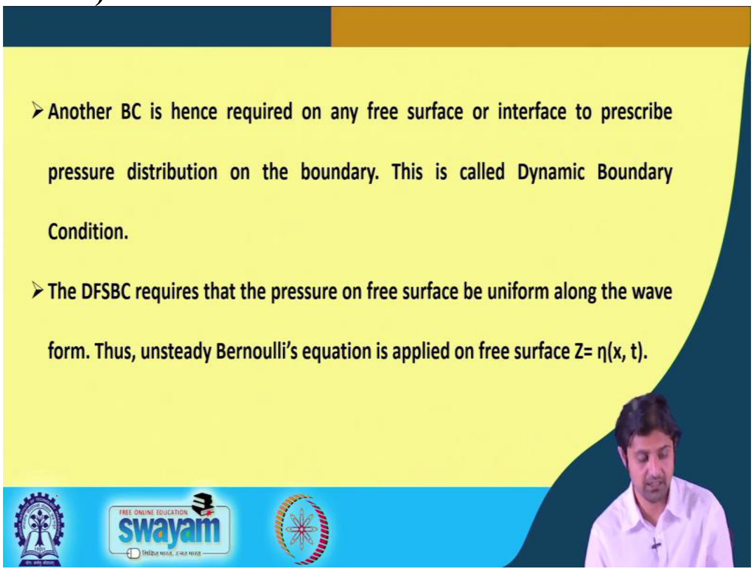

Therefore another boundary conditions is required for any free surface or interface to prescribe the pressure distribution and the boundary. Since the free surface cannot support the pressure variation, we need to prescribe another boundary condition to tell what the pressure distribution on that particular boundary is and this is called the dynamic boundary condition. So, you see, we started from the boundary conditions, we went to the dynamic boundary condition where we saw the bottom boundary condition and kind emetic free surface boundary condition.

But 1 typical thing with 3 surfaces are they cannot support pressure variations across it, unlike the fixed surface, therefore another boundary condition or bottom bound another boundary condition is required that is called the dynamic boundary condition. Dynamic free surface boundary condition requires that the requirement of the dynamic 3 surface boundary condition is that pressure on the free surface be uniform along the way form.

So, along that free surface distortion or wave as we call it, the requirement is that the pressure on the free surface should be uniform along the way form. And here to derive that we use unsteady Bernoulli's equation on the free surface. So, we have seen the Bernoulli's equation for this we are going to use unsteady Bernoulli's equation unsteady means there is going to be another term del t.

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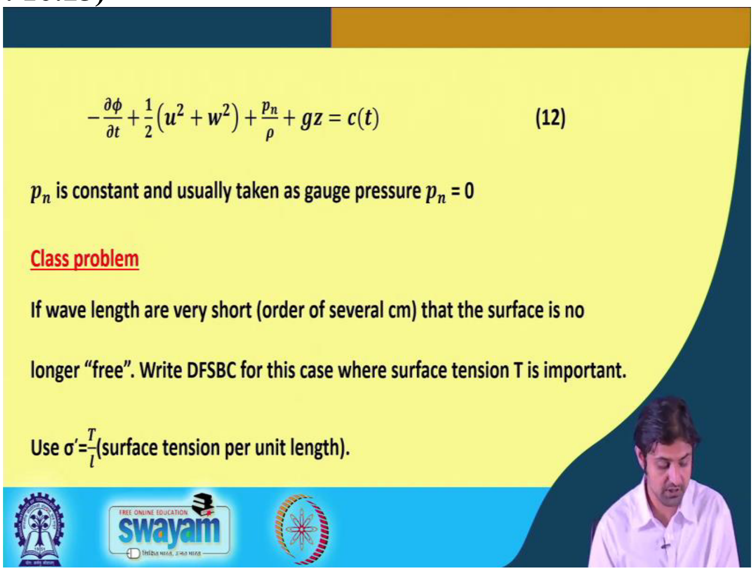

So, you remember this was a typical Bernoulli's equation to make it unsteady we add another term this is unsteady part. So, to derive the dynamic free surface boundary condition we use when we make use of the unsteady Bernoulli's equation here rho n is constant and usually taken as gauge pressure that is 0 atmospheric pressure, but we take it Pn = 0. Now, if there is a wavelength of very I mean in wavelength or very short that the surface is no longer free.

You have to ride the dynamic free surface boundary condition for this case where surface tension tease important. So, to be able to do that, you know, I will solve the dynamic 3 surface boundary condition in this case.

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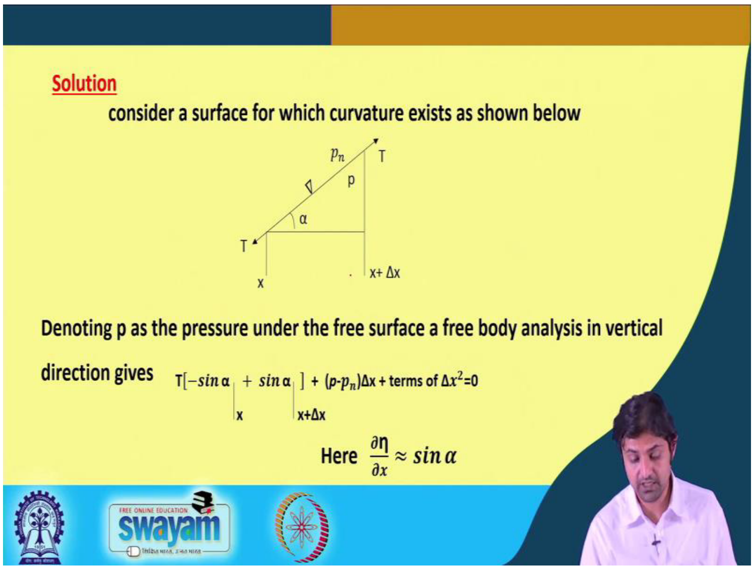

Considered as a surface for which the curvature exists n is as shown below here all. So, this is T this is T and this is alpha and the gauge pressure is Pn and we have considered this distance delta x all. So, if you denote P as the pressure under the free surface and free body analysis in vertical direction, so, we have to do the free body analysis in the vertical direction. So, this is the direction vertical direction v direction. So, we can write t - sine alpha because this will have component here and sine alpha + sine alpha at this point, you know.

Because this is negative this will be positive + because of this pressure difference P - P n into delta x + different terms of delta x is squared. So, here we assume that delta eta by delta x = sine alpha all how you see this is so, this will be at this point there will be eta and there will be point eta. So, the difference between them del eta by del x will be sine alpha all or we can take this 1 also as delta x because we are considering very small. So, here we take delta eta by delta x = sine alpha.

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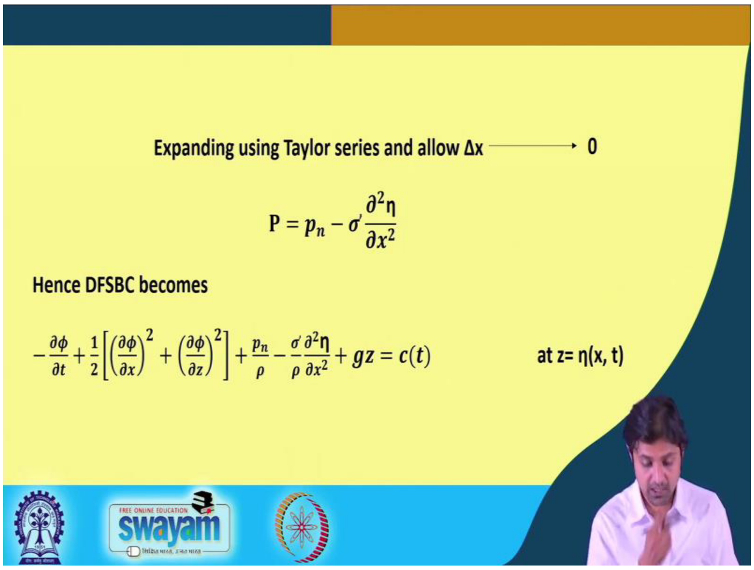

So, if we expand it using the Taylor series and allow delta x going to 0, we will get P = P n - sigma dash is nothing but T del square eta by del x squared, if you put sin a del eta del x at sin alpha this is what we are going to get. So, therefore, the dynamic 3 surface boundary condition becomes. So, what we have done we have just prescribed Pn you see P total pressure therefore, it becomes Pn by rho -. So, instead of P here in the dynamic preface boundary condition.

We find put the pressure here and the rest of the things remains same you see, so, this is u squared this is v squared + P by rho. So, that becomes Pn by rho - sigma dash or T by rho into Del squared eta by del x squared + z = c of t. This was a very simple example. You must have prior knowledge of the surface tension from before to be able to do that how to apply this problem, but this is just a way of showing the dynamic face surface boundary condition.

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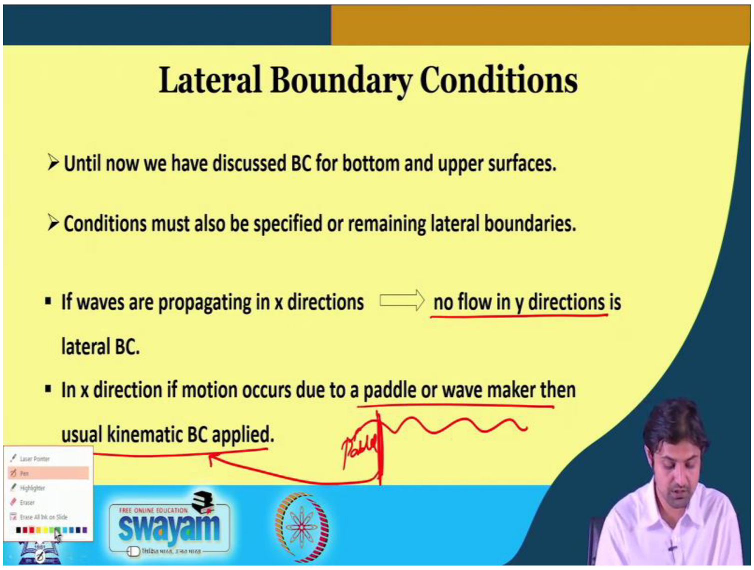

Now, another boundary condition that is called lateral boundary condition. Until now, we have discussed bottom boundary condition for bottom and upper surfaces. So, bottom boundary condition the at the top we have discussed about kinematic free surface boundary condition and dynamic free surface boundary condition, but the but there are many other surfaces this is also one of the boundary. So, these we must also specify the remaining lateral boundary conditions.

So, let us say one of the conditions could be if waves are propagating in x direction that means, by the sentence itself this means there is no flow in y direction. Therefore, the no flow in y direction is 1 lateral boundary condition. In next direction if motion occurs due to a paddle or the wave maker then we can apply the usual kinematic boundary condition on this side. So, the waves are being generated here. So due to a paddle here so, the boundary condition at this surface we can apply similar to like the kinematic boundary condition no changes.

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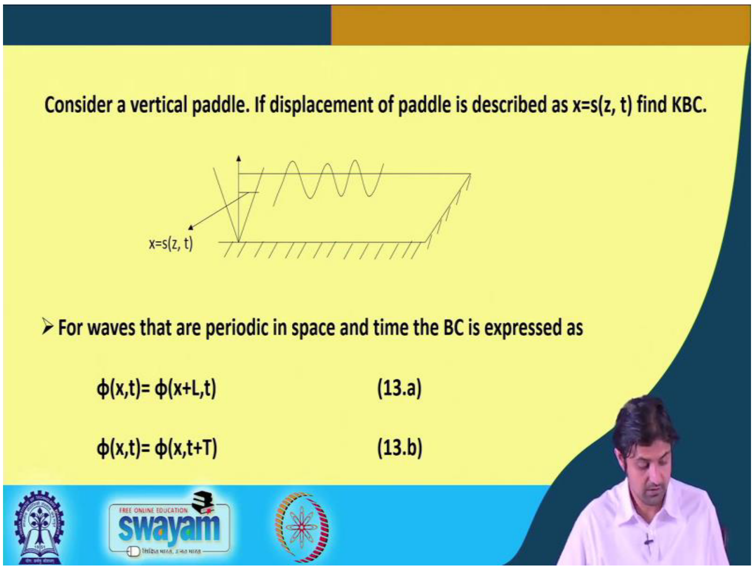

So, consider a vertical paddle if the displacement of the paddle is described as x is a function of s z, t the question is find the kinetic boundary condition. So, this is x is a function of z, t this is the paddle, so, the movement is the next direction of the paddle therefore, we say x = a function of z, t so with this surface equation x - s z, t = 0 are F you can use the similar analysis of the kinematic boundary condition.

you remember u dot n = - del x del t by mod delta F all the same thing you will get another lateral boundary condition could be for the waves that are periodic means, they have a wave period and it repeats in space and time. So, for the waves that are periodic in space and time the boundary condition can be simply represented as, phi as phi of x, t will be same as x + L where L is the wavelength.

This is periodic boundary condition in space, the other could be it will be same as at t + delta T. So, this capital T is wave period you would get to know what wavelength and wave period are soon, but to complete this boundary condition description, I think this is quite an important thing.

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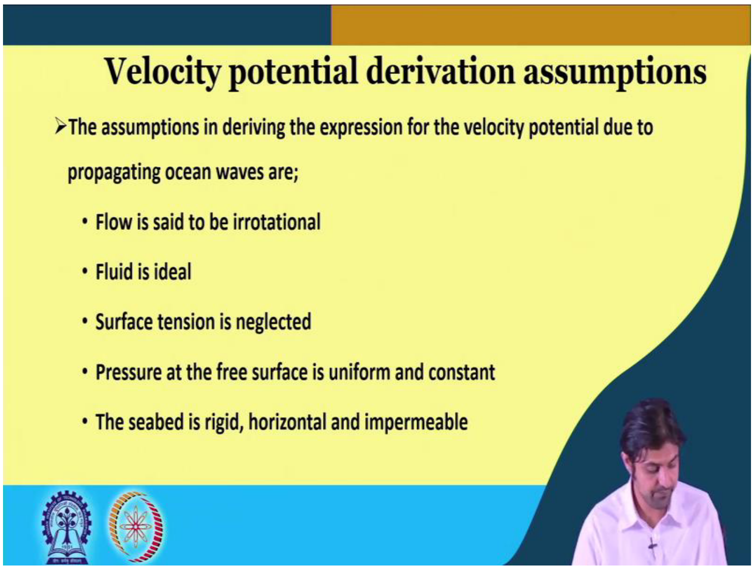

Now, after reading the boundary condition, we are getting straight to the derivation of the velocity potential to derive the velocity potential there are some assumptions, we started this chapter I mean this module by saying that it is irrotational flow. So, following are the assumptions of deriving the expression for the velocity potential phi that has much to do with the hydraulic engineering or fluid mechanics.

And it could be due to anything in our current case that are due to the propagating ocean waves are that the flow is irrotational all, fluid is ideal surface tension is also neglected and the pressure at free surfaces uniform and constant also the seabed is rigid horizontal and impaired may well these are some of the top most top 5 condition and this ideal in inviscid flow total ideal conditions we have assumed all.

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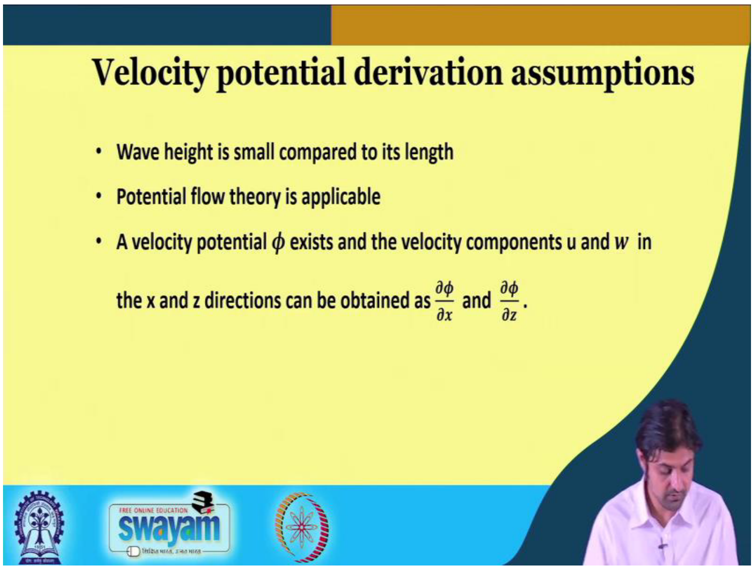

Now, we also assume some additional conditions that wave height is small compared to the length. So, we see yes, you see. So, this is let us say amplitude = wave height by 2 for this wavelength lambda by 2. So, we say this lambda by 2 is much larger than h by 2 this is not a very good representation, but just to let you know that we will follow this derivation we will assume wave height is small compared to its length we will come to why the potential flow theory is applicable, because we have assumed the ideal fluid.

Therefore, under the ideal conditions like this a velocity potential phi exists and the velocity component u and w in the x and z directions can be obtained as del phi del x and del phi del z. So, we have assumed that the flow is occurring into direction x and z therefore, there will be only 2 velocity components, u and w and if we know phi we can obtain u as del phi del x and w as del phi del y.

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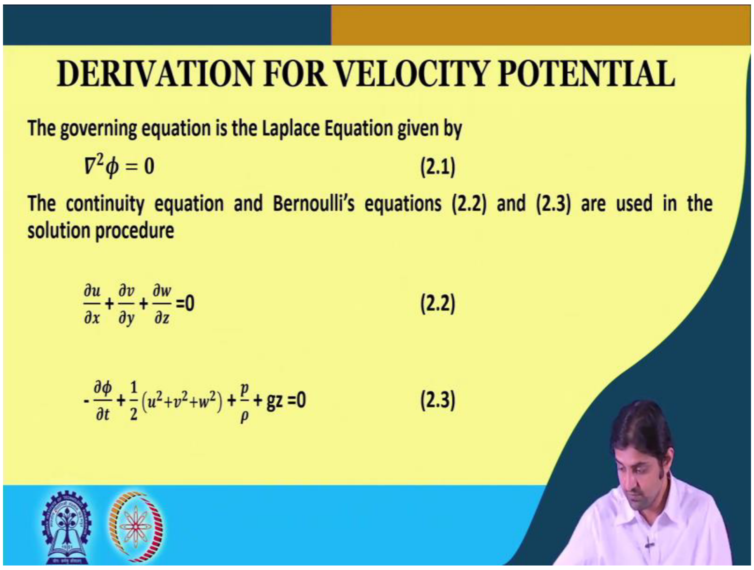

So, the governing equation so we do go back and remember what we learned in the Laplace equation the Laplace equation was we go back and listen to remember try to remember the boundary conditions. There we said first we establish a region of interest that we have already done ocean surface waves or oceans are the region of interest. Now, we must find the equation that satisfy that should be satisfied, because of the assumptions we have taken.

In the current scenario, we can assume that Laplace equation is that governing equation which will be satisfied. So Laplace equation for phi can be return as delta squared phi = 0 for our current case. The continuity equation and the Bernoulli's equation are used in the solution procedure. So this is the continuity equation del u del x + del v del y + delta w delta z = 0 and also the unsteady Bernoulli's equation for the boundary condition,

You remember the dynamic resurface boundary condition. So, these are the 3 equations that are used for the solution procedure all.

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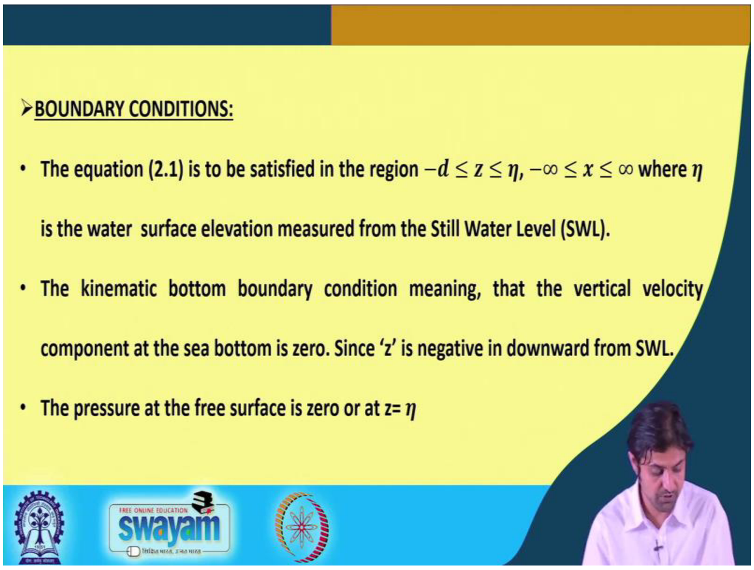

Now, the boundary conditions equation 2.1 you know this Laplace equation is to be satisfied from the water depth if we say the water depth is - d to the free surface of the wave. So, try to I mean you see. So, origin is at free surface so, this should be satisfied. This is eta and this is z should be satisfied from - d to eta and we have considered the entire ocean so x will be - infinity to + infinity. Here eta is the water surface elevation measured from Stillwater level if this is a still water level.

So, this is a top because below is the depth here is the eta, eta is the water surface elevation measured from still water level this is SWL, the kinematic boundary condition meaning that the vertical velocity component at the sea bottom is 0 because at the sea bottom the we already seen that w should be 0. This is one of the condition and the pressure at free surface is 0 atmospheric pressure but we have assumed it to be gauged pressure Pn = 0 at z = eta, these are the boundary conditions.

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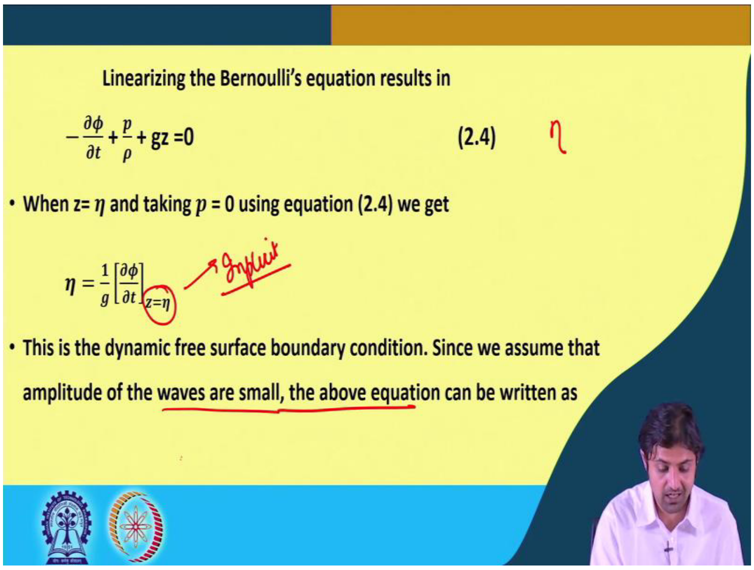

Now, if you linearize the Bernoulli's equation, so linearizing means taking out the second order term. So, we take out u squared + w squared because those are quite small at the free surface. So, this is the linearized boundary condition that we linear linearized Bernoulli's equation that we get.

So, when the height will be z = eta that means, that free surface if we take phi = 0 equation in equation 2.4 we are going to get eta = 1 by g del phi del t at z = eta. So, in terms of velocity potential we have obtained by linearizing the Bernoulli's equation we have obtained you see why we have assumed u squared + w squared = 0. You see at the top there will be no velocity of the fluid particle.

That is why it is going down under the influence of gravity that is why half u squared + W squared is going to 0 all and that is why we were able to linearize and avoid this term in here. So, this is the eta = So, eta which was unknown from before using the dynamic 3 surface boundary condition this dynamic 3 surface boundary condition, we have been able to write eta as a function of velocity potential.

You see how important that tool was so, this is the dynamic free surface boundary condition since we assume that the amplitude of the wave are small. See, 1 assumption was that the amplitude has amplitude of the waves is very small. So this eta actually is very small. So we can also this is a problem if it says z = eta because we do not know eta from before this becomes implicit equation. So with this assumption of that the waves being as small.

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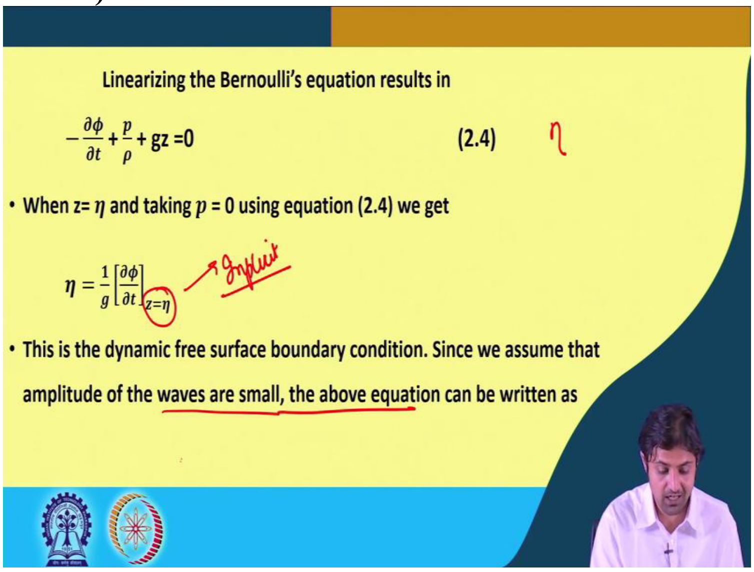

This data can be written as, instead of 0 = eta we write z = 0 because eta is very small, so, it will be z = 0. Now, you see this is an explicit equation in eta of eta, where if we know the velocity potential, we will be able to find and found by assumption of small amplitude. So, this has been founded by assumption of a small amplitude. So, this is applicable only when eta is small and valid for these condition which you do not have to worry about it now. So, I will show you the definition sketch.

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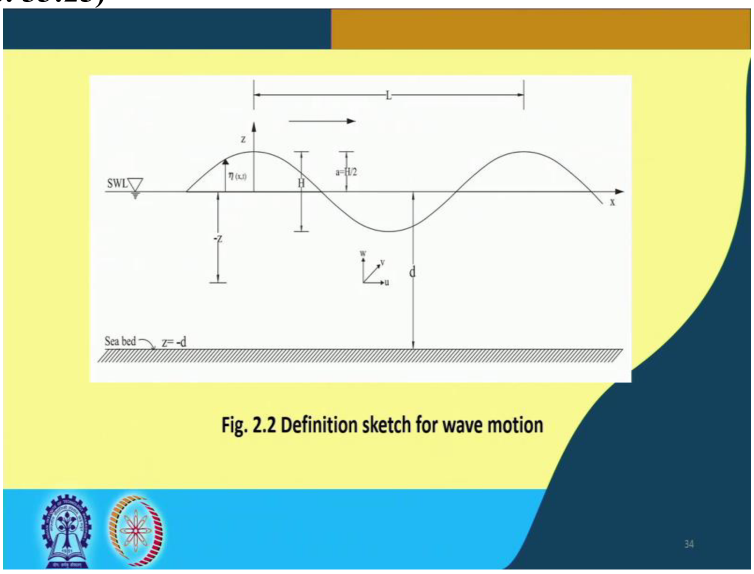

So, just describing see this is the sea bed z = - d this is still water level z is 0 at this line. So, this is the depth - d. So, this is the half of wave height or amplitude the full way will be between the crest and rough whatever the height is wavelength will be from this point to this point,. So, I think this is a nice point to stop before we meet again for the next lecture, where we start again by describing very basic properties of this basic sketch of the way of motion, thank you so much. See you in the next lecture.