**Solid Mechanics**
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**Lecture - 17**
**Linear Momentum Balance in Cylindrical Coordinate System**

Hello everyone! Welcome to Lecture 17! In this lecture, we will derive the Linear Momentum Balance in cylindrical coordinate system.

## 1 Cylindrical Coordinate System (start time: 00:27)

Cylindrical coordinate system turns out to be very handy in studying deformation of bodies having cylindrical shape as we will see later. It is therefore of importance to derive balance laws in cylindrical coordinate system. Figure 1a shows a generalized cylinder whose radius may change over the axis of the cylinder. The axis of the cylinder coincides with $z$ axis. Any general point of the cylinder will have coordinates $(r,\theta,z)$. The position vector of this point is $r\underline{e}_r + z\underline{e}_z$ where the $\theta$ dependence is hidden in the basis vector $\underline{e}_r$. If we project this point on $z = 0$ plane, we get to $(r,\theta,0)$ as shown in Figure 1a.

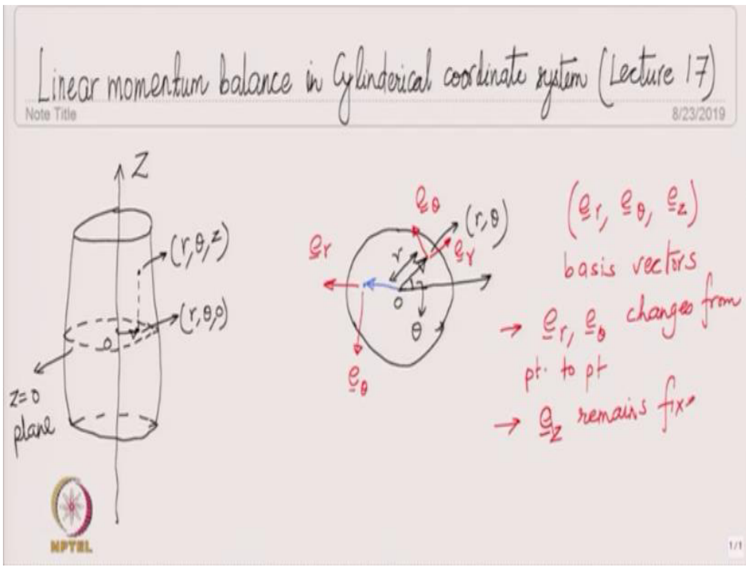


Figure 1: (a) A generalized cylinder with its axis coinciding with $z$ axis (b) $z = 0$ plane

### 1.1 Basis Vectors(start time: 02:44)

We have drawn the $z = 0$ plane in Figure 1b. There is a baseline or a reference line relative to which angle $\theta$ is measured. The length of the line joining the center and the projected point is $r$ while the angle that it makes from the baseline is $\theta$. Two of the basis vectors lie in this plane: $\underline{e}_r$ points radially outward from the center while $\underline{e}_\theta$ points in the direction of increasing $\theta$ and is perpendicular to $\underline{e}_r$. The third basis vector $\underline{e}_z$ lies along the axis of the cylinder. Thus, the basis vectors for a cylindrical coordinate system are $(\underline{e}_r,\underline{e}_\theta,\underline{e}_z)$ whereas in the Cartesian coordinate system, the basis vectors are $(\underline{e}_1,\underline{e}_2,\underline{e}_3)$. Although both the sets of basis vectors are orthonormal, there is a big difference in their properties: the basis vectors of Cartesian coordinate system are fixed in direction and do not change from one point to the other but in cylindrical coordinate system, two of the basis vectors ($\underline{e}_r, \underline{e}_\theta$) change when $\theta$ coordinate of a point changes. This has been illustrated in Figure 2a where the basis vectors at the two points are differently oriented although only the $\theta$ coordinate of the two points is different. The third basis vector $\underline{e}_z$ remains fixed at every point though. One thing to note here is that $\underline{e}_r$ and $\underline{e}_\theta$ change only when the $\theta$ coordinate changes. If only the radial coordinate or/and z-coordinate of a point is changed, the basis vectors do not change (see Figure 2b).

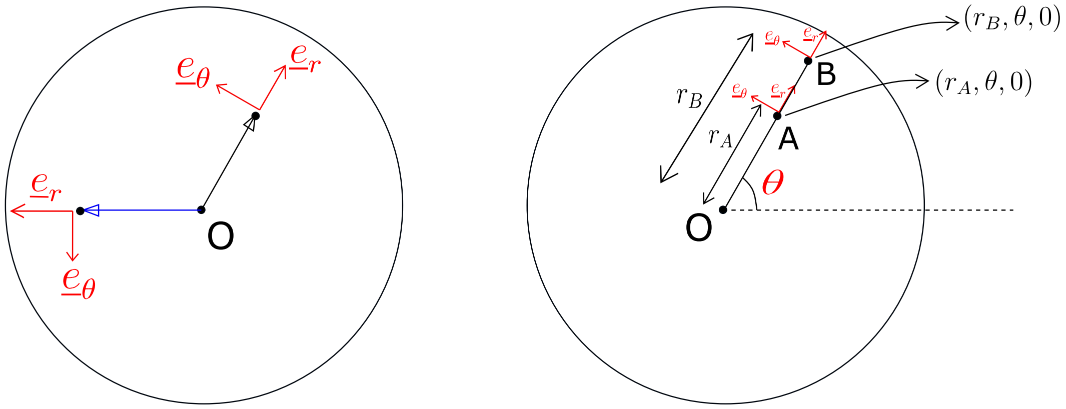


Figure 2: (a) Two points are shown on $z = 0$ plane along with basis vectors at those points (b) Two points A and B are shown on $z = 0$ plane along the same radial line and the basis vectors at the two points are shown to be the same

### 1.2 Cylindrical element (start time: 08:36)

In Cartesian coordinate system, we analyzed cuboid elements with their face normals directed along the coordinate axis. In cylindrical coordinate system similarly, we need to analyze cylindrical elements with their face normals parallel to $\underline{e}_r$, $\underline{e}_\theta$ and $\underline{e}_z$. A cylindrical element for the generalized cylinder and its zoomed view is shown in Figure 3. The top face has its normal along $z$ axis. Thus, the top face is $+z$ plane and the bottom face is $-z$ plane. The front face shown has its normal along $-\underline{e}_\theta$ direction. Thus, this is the $-\theta$ plane and the plane opposite to it is $+\theta$ plane. Finally, the concave curved face has its plane normal along $-\underline{e}_r$ direction and can be imagined to become flat in the limit of the cylindrical element becoming infinitesimally small. Likewise, the convex curved face is the $+r$ plane.

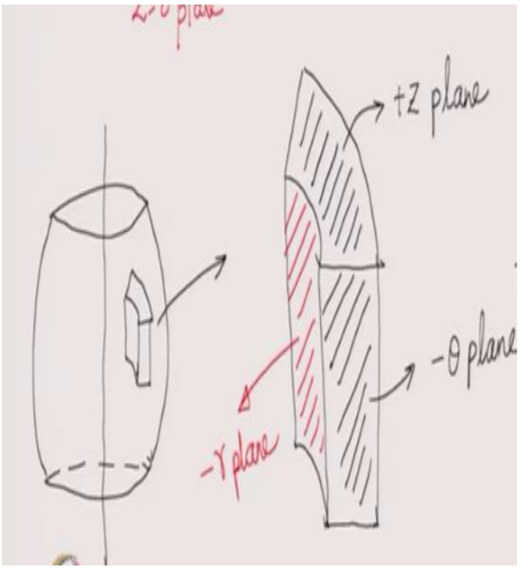


Figure 3: (a) A small cylindrical element in a generalized cylinder (b) a zoomed view of the cylindrical element with its various faces shown

## 2 LMB Formulation (start time: 16:26)

For the Linear Momentum Balance, we need to sum the total force on the cylindrical element due to tractions and body force and equate it to its rate of change of linear momentum. Let us begin with considering each of the faces separately, obtain total force on them and eventually add all of them. In Figure 4a, we have shown the traction components (resolved along the three basis vectors) acting on each of these faces from the remaining part of the body.

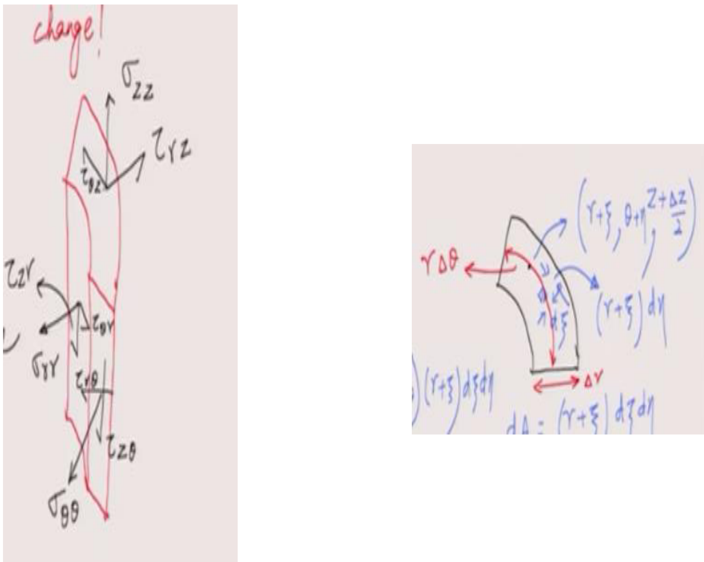

Figure 4: (a) Decomposition of traction vector along the three basis directions for various faces of the cylindrical element (b) the +$z$ face of the cylindrical element and various lengths associated with it: an infinitesimal area element on this face is also shown

## 2.1 Force on +$z$ and −$z$ plane (start time: 17:06)

On +$z$ plane, the traction vector ($\underline{t}^{+z}$) is given by

$$\underline{t}^{+z} = \sigma_{zz}\underline{e}_z + \tau_{rz}\underline{e}_r + \tau_{\theta z}\underline{e}_\theta \tag{1}$$

The total force due to this traction can be obtained by integrating it over +$z$ plane which we have drawn separately in Figure 4b. The difference between the outer and inner radius of this face is $\Delta r$ and the total angle subtended by the face at the center is $\Delta\theta$. The center of the cylindrical element shown in Figure 3 has coordinates ($r,\theta,z$). Thus, the coordinates of a general point on the top surface is denoted by ($r + \xi$, $\theta + \eta$, $z + \frac{\Delta z}{2}$) ($r$ and $\theta$ coordinates vary while the $z$ coordinate remains fixed). We then consider an infinitesimal area element on this face. The length of this area element in the radial direction is $d\xi$ and the width in the $\vartheta$ direction is $r$-coordinate multiplied with $d\eta$. Thus, the area of the infinitesimal element is:

$$dA = l \times b = d\xi \times (r + \xi)d\eta = (r + \xi)d\xi d\eta \tag{2}$$

Hence, the total force on +$z$ plane is:

$$\underline{F}^{+z} = \iint_A \underline{t}^{+z} dA = \int_{\xi=\frac{-\Delta r}{2}}^{\frac{\Delta r}{2}} \int_{\eta=\frac{-\Delta\theta}{2}}^{\frac{\Delta\theta}{2}} \underbrace{\Big(\sigma_{zz}\underline{e}_z + \tau_{rz}\underline{e}_r + \tau_{\theta z}\underline{e}_\theta\Big)}_{\text{at } (r+\xi,\theta+\eta,z+\frac{\Delta z}{2})}(r+\xi)d\xi d\eta \tag{3}$$

For $-z$ plane, all the traction components act along negative basis directions but an arbitrary point on $-z$ plane will have coordinates ($r+\xi$, $\theta+\eta$, $z-\frac{\Delta z}{2}$). Thus, the total force on $-z$ plane will be

$$\underline{F}^{-z} = \iint_A \underline{t}^{-z} dA = -\int_{\xi=\frac{-\Delta r}{2}}^{\frac{\Delta r}{2}} \int_{\eta=\frac{-\Delta\theta}{2}}^{\frac{\Delta\theta}{2}} \underbrace{\left(\sigma_{zz}\underline{e}_z + \tau_{rz}\underline{e}_r + \tau_{\theta z}\underline{e}_\theta\right)}_{\text{at } (r+\xi,\theta+\eta,z-\frac{\Delta z}{2})}(r+\xi)d\xi d\eta \tag{4}$$

### 2.2 Taylor's series expansion (start time: 25:48)

As the stress components are varying in the domain of integration, we can use Taylor's expansion about the center of the cylindrical element, i.e., at ($r,\theta,z$). In equation (3), e.g, we need to Taylor expand both the stress components as well as basis vectors. Ignoring higher order terms, we get

$$\begin{aligned}\underline{F}^{+z} = \; & \textstyle\iint_A \left[\sigma_{zz} + \frac{\partial\sigma_{zz}}{\partial r}\xi + \frac{\partial\sigma_{zz}}{\partial\theta}\eta + \frac{\partial\sigma_{zz}}{\partial z}\frac{\Delta z}{2}\right]\underline{e}_z dA \\ & + \left[\tau_{rz} + \frac{\partial\tau_{rz}}{\partial r}\xi + \frac{\partial\tau_{rz}}{\partial\theta}\eta + \frac{\partial\tau_{rz}}{\partial z}\frac{\Delta z}{2}\right]\left(\underline{e}_r + \frac{\partial\underline{e}_r}{\partial r}\xi + \frac{\partial\underline{e}_r}{\partial\theta}\eta + \frac{\partial\underline{e}_r}{\partial z}\frac{\Delta z}{2}\right)dA \\ & + \left[\tau_{\theta z} + \frac{\partial\tau_{\theta z}}{\partial r}\xi + \frac{\partial\tau_{\theta z}}{\partial\theta}\eta + \frac{\partial\tau_{\theta z}}{\partial z}\frac{\Delta z}{2}\right]\left(\underline{e}_\theta + \frac{\partial\underline{e}_\theta}{\partial r}\xi + \frac{\partial\underline{e}_\theta}{\partial\theta}\eta + \frac{\partial\underline{e}_\theta}{\partial z}\frac{\Delta z}{2}\right)dA.\end{aligned} \tag{5}$$

As $\underline{e}_r$ and $\underline{e}_\theta$ change only with $\theta$, their partial derivatives with respect to $r$ and $z$ will be zero leading to

$$\begin{aligned}\underline{F}^{+z} = \iint \; & \left[\sigma_{zz} + \frac{\partial\sigma_{zz}}{\partial r}\xi + \frac{\partial\sigma_{zz}}{\partial\theta}\eta + \frac{\partial\sigma_{zz}}{\partial z}\frac{\Delta z}{2}\right]\underline{e}_z dA \\ & + \left[\tau_{rz} + \frac{\partial\tau_{rz}}{\partial r}\xi + \frac{\partial\tau_{rz}}{\partial\theta}\eta + \frac{\partial\tau_{rz}}{\partial z}\frac{\Delta z}{2}\right]\left(\underline{e}_r + \frac{\partial\underline{e}_r}{\partial\theta}\eta\right)dA \\ & + \left[\tau_{\theta z} + \frac{\partial\tau_{\theta z}}{\partial r}\xi + \frac{\partial\tau_{\theta z}}{\partial\theta}\eta + \frac{\partial\tau_{\theta z}}{\partial z}\frac{\Delta z}{2}\right]\left(\underline{e}_\theta + \frac{\partial\underline{e}_\theta}{\partial\theta}\eta\right)dA\end{aligned} \tag{6}$$

### 2.3 Partial derivative of basis vectors w.r.t. $\vartheta$ (start time: 31:20)

We now show how to obtain $\frac{\partial\underline{e}_r}{\partial\theta}$ and $\frac{\partial\underline{e}_\theta}{\partial\theta}$. We again draw $z=0$ plane as shown in Figure 5a. The $\theta$ coordinates of the two points A and B differ by $\Delta\theta$. Using the definition of differentiation, we can write

$$\frac{\partial\underline{e}_r}{\partial\theta} = \lim_{\Delta\theta\to 0}\frac{\underline{e}_r(\theta+\Delta\theta) - \underline{e}_r(\theta)}{\Delta\theta} \tag{7}$$

To obtain it geometrically, we parallely transport the two radial vectors such that they start from a common point as shown in Figure 5b.

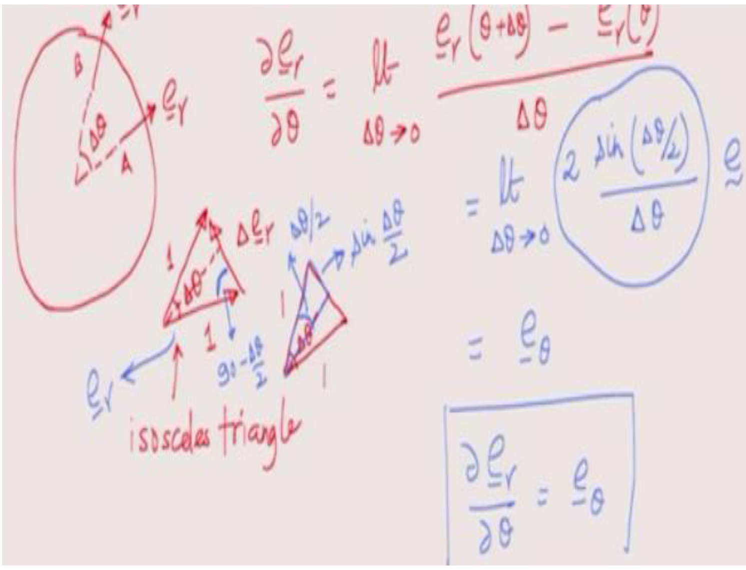


Figure 5: (a) Two points A and B are shown on $z$ = 0 plane along with $\underline{e}_r$ at the two points: their $\theta$ coordinates differ by $\Delta\theta$ (b) the two radial vectors at points A and B are translated so that they start from a common point

The angle between the two radial vectors is still $\Delta\theta$. The numerator in equation (7) will thus equal the third side of the triangle in Figure 5b which is also denoted by $\Delta\underline{e}_r$ there. The magnitude of the two sides representing the basis vectors has magnitude 1 as they represent unit vectors. We then draw an angle bisector to the angle $\Delta\theta$. As the original triangle is an isosceles triangle, the angle bisector is also the perpendicular bisector to the base. Let us then consider any one of the two right triangles: the magnitude of its base equals $\sin(\frac{\Delta\theta}{2})$. So, the total magnitude of $\Delta\underline{e}_r$ will be $2\sin(\frac{\Delta\theta}{2})$. Let us denote the direction of this vector by $\underline{e}$. Substituting these results in equation (7), we get

$$\frac{\partial \underline{e}_r}{\partial \theta} = \lim_{\Delta\theta \to 0} \frac{2\sin(\frac{\Delta\theta}{2})}{\Delta\theta}\underline{e} \tag{8}$$

The direction $\underline{e}$ makes an angle of $90^\circ - \frac{\Delta\theta}{2}$ with $\underline{e}_r$ as shown in Figure 5b. In the limit of $\Delta\theta \to 0$, this angle becomes $90^\circ$. Thus, in this limit, $\Delta\underline{e}_r$ points along $\underline{e}_\theta$. Also, the magnitude term in equation (8) becomes 1 ($\because \lim_{x\to 0} \frac{\sin x}{x} = 1$). Thus, we finally have

$$\frac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta \tag{9}$$

By a similar derivation, we can show that

$$\frac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r \tag{10}$$

## 2.4 Final Simplification (start time: 38:18)

We can now plug in the derivatives of basis vectors into equation (6) to get

$$\begin{aligned}\underline{F}^{+z} = \iint \quad & \left[\sigma_{zz} + \frac{\partial \sigma_{zz}}{\partial r}\xi + \frac{\partial \sigma_{zz}}{\partial \theta}\eta + \frac{\partial \sigma_{zz}}{\partial z}\frac{\Delta z}{2}\right]\underline{e}_z dA \\ + & \left[\tau_{rz} + \frac{\partial \tau_{rz}}{\partial r}\xi + \frac{\partial \tau_{rz}}{\partial \theta}\eta + \frac{\partial \tau_{rz}}{\partial z}\frac{\Delta z}{2}\right]\left(\underline{e}_r + \eta \underline{e}_\theta\right) dA \\ + & \left[\tau_{\theta z} + \frac{\partial \tau_{\theta z}}{\partial r}\xi + \frac{\partial \tau_{\theta z}}{\partial \theta}\eta + \frac{\partial \tau_{\theta z}}{\partial z}\frac{\Delta z}{2}\right]\left(\underline{e}_\theta - \eta \underline{e}_r\right) dA.\end{aligned} \tag{11}$$

Let us now consider the first set of terms, i.e.,

$$\iint \left[\sigma_{zz} + \frac{\partial \sigma_{zz}}{\partial r}\xi + \frac{\partial \sigma_{zz}}{\partial \theta}\eta + \frac{\partial \sigma_{zz}}{\partial z}\frac{\Delta z}{2}\right]\underline{e}_z dA. \tag{12}$$

In the first term here, $\sigma_{zz}$ is is evaluated at the center of the cylindrical element and hence can be taken out of the integral. The integral of $dA$ then gives us $A_z$: the area of $+z$ face. Similar logic applies to the last term here. Considering the second term, the integration of $\xi dA$ will give us $A_z$ times $r_{\text{centroid}}$ of $+z$ face. As we are measuring $\xi$ from the centroid itself, $r_{\text{centroid}}$ becomes zero. Likewise, the integration of $\eta dA$ in the thirs term will be zero. Thus, we have

$$\iint \left[\sigma_{zz} + \frac{\partial \sigma_{zz}}{\partial r}\xi + \frac{\partial \sigma_{zz}}{\partial \theta}\eta + \frac{\partial \sigma_{zz}}{\partial z}\frac{\Delta z}{2}\right]\underline{e}_z dA = \left[\sigma_{zz}A_z + \frac{\partial \sigma_{zz}}{\partial z}\frac{\Delta z}{2}A_z\right]\underline{e}_z = \left[\sigma_{zz}A_z + \frac{\partial \sigma_{zz}}{\partial z}\frac{\Delta V}{2}\right]\underline{e}_z \tag{13}$$

Here $\frac{\Delta V}{2}$ denotes half the volume of the cylindrical element. Now, consider the second set of terms in equation (11):

$$\iint \left[\tau_{rz} + \frac{\partial \tau_{rz}}{\partial r}\xi + \frac{\partial \tau_{rz}}{\partial \theta}\eta + \frac{\partial \tau_{rz}}{\partial z}\frac{\Delta z}{2}\right]\left(\underline{e}_r + \eta \underline{e}_\theta\right) dA. \tag{14}$$

The integration of the big bracket multiplied with $\underline{e}_r$ will give us $\left[\tau_{rz}A_z + \frac{\partial \tau_{rz}}{\partial z}\frac{\Delta V}{2}\right]\underline{e}_r$ by a similar analysis as used for the first set of terms. Let us now look at multiplication of the big bracket with the second term. The terms $\tau_{rz}$, its partial derivatives and $\underline{e}_\theta$ are constant in the domain of integration as they are evaluated at the center of the cylindrical element. Thus, the first term will be $\tau_{rz}\underline{e}_\theta \iint \eta dA$ which will be zero again. For the second term, we will get

$$\begin{aligned}
& \frac{\partial \tau_{rz}}{\partial r}\underline{e}_\theta \iint_A \xi\eta dA \\
= & \frac{\partial \tau_{rz}}{\partial r}\underline{e}_\theta \iint \xi\eta(r+\xi)d\xi d\eta \qquad \text{(using (2))} \\
= & \frac{\partial \tau_{rz}}{\partial r}\underline{e}_\theta \int_{-\frac{\Delta r}{2}}^{\frac{\Delta r}{2}} \xi(r+\xi)d\xi \int_{-\frac{\Delta\theta}{2}}^{\frac{\Delta\theta}{2}} \eta d\eta \\
= & \frac{\partial \tau_{rz}}{\partial r}\underline{e}_\theta \int_{-\frac{\Delta r}{2}}^{\frac{\Delta r}{2}} \xi(r+\xi)d\xi \left[\frac{\eta^2}{2}\right]\Bigg|_{-\frac{\Delta\theta}{2}}^{\frac{\Delta\theta}{2}} \\
= & 0
\end{aligned} \tag{15}$$

The third term is $\frac{\partial \tau_{rz}}{\partial \theta}\underline{e}_\theta \iint \eta^2 dA$. When we work this out, we will get something that is of the order smaller than the volume of the cylindrical element, i.e., $o(\Delta V)$. This is because both $\eta^2$ and $dA$ terms have dimension squares of the length. Thus the integral will be fourth power of the length dimension. The last term $\frac{\partial \tau_{rz}}{\partial z}\underline{e}_\theta \frac{\Delta z}{2} \iint \eta dA$ will also become zero. Thus, the second set of terms in (11) finally gives us

$$\iint \left[\tau_{rz} + \frac{\partial \tau_{rz}}{\partial r}\xi + \frac{\partial \tau_{rz}}{\partial \theta}\eta + \frac{\partial \tau_{rz}}{\partial z}\frac{\Delta z}{2}\right]\left(\underline{e}_r + \eta\underline{e}_\theta\right) dA = \left[\tau_{rz}A_z + \frac{\partial \tau_{rz}}{\partial z}\frac{\Delta V}{2}\right]\underline{e}_r + o(\Delta V) \tag{16}$$

The last set of terms can be similarly simplified to yield

$$\iint \left[\tau_{\theta z} + \frac{\partial \tau_{\theta z}}{\partial r}\xi + \frac{\partial \tau_{\theta z}}{\partial \theta}\eta + \frac{\partial \tau_{\theta z}}{\partial z}\frac{\Delta z}{2}\right]\left(\underline{e}_\theta - \eta\underline{e}_r\right) dA = \left[\tau_{\theta z}A_z + \frac{\partial \tau_{\theta z}}{\partial z}\frac{\Delta V}{2}\right]\underline{e}_\theta + o(\Delta V) \tag{17}$$

The total force on +*z* plane will thus be

$$\underline{F}^{+z} = \left(\sigma_{zz}A_z + \frac{\partial \sigma_{zz}}{\partial z}\frac{\Delta V}{2}\right)\underline{e}_z + \left(\tau_{rz}A_z + \frac{\partial \tau_{rz}}{\partial z}\frac{\Delta V}{2}\right)\underline{e}_r + \left(\tau_{\theta z}A_z + \frac{\partial \tau_{\theta z}}{\partial z}\frac{\Delta V}{2}\right)\underline{e}_\theta + o(\Delta V) \tag{18}$$

We can also find the force on −*z* plane in the same way. If we compare equations (3) and (4), we find that we have an overall negative sign for the −*z* plane and also the stress components are to be evaluated at $z - \frac{\Delta z}{2}$ instead of $z + \frac{\Delta z}{2}$. So, whenever we will have a term of the form $\frac{\partial}{\partial z}$, it will be multiplied with $\frac{-\Delta z}{2}$. The overall negative sign will then make such terms positive while the other terms will become negative. Thus, we will get

$$\underline{F}^{-z} = \left(-\sigma_{zz}A_z + \frac{\partial \sigma_{zz}}{\partial z}\frac{\Delta V}{2}\right)\underline{e}_z + \left(-\tau_{rz}A_z + \frac{\partial \tau_{rz}}{\partial z}\frac{\Delta V}{2}\right)\underline{e}_r + \left(-\tau_{\theta z}A_z + \frac{\partial \tau_{\theta z}}{\partial z}\frac{\Delta V}{2}\right)\underline{e}_\theta + o(\Delta V) \tag{19}$$

Upon adding the two equations (18) and (19), we finally get

$$\underline{F}^{+z} + \underline{F}^{-z} = \left( \frac{\partial \sigma_{zz}}{\partial z} \underline{e}_z + \frac{\partial \tau_{rz}}{\partial z} \underline{e}_r + \frac{\partial \tau_{\theta z}}{\partial z} \underline{e}_\theta \right) \Delta V + o(\Delta V) \tag{20}$$

We will work out the remaining derivation in the next lecture.