Chapter 28

ELEMENTS of STRUCTURAL RELIABILITY

28.1 Introduction

1 Traditionally, evaluations of structural adequacy have been expressed by safety factors $SF = \frac{C}{D}$, where C is the *capacity* (i.e. strength) and D is the demand (i.e. load). Whereas this evaluation is quite simple to understand, it suffers from many limitations: it 1) treats all loads equally; 2) does not differentiate between capacity and demands respective uncertainties; 3) is restricted to service loads; and last but not least 4) does not allow comparison of relative reliabilities among different structures for different performance modes. Another major deficiency is that all parameters are assigned a single value in an analysis which is then *deterministic*.

2 Another approach, a *probabilistic* one, extends the factor of safety concept to explicitly incorporate uncertainties in the parameters. The uncertainties are quantified through statistical analysis of existing data or judgmentally assigned.

3 This chapter will thus develop a procedure which will enable the Engineer to perform a *reliability* based analysis of a structure, which will ultimately yield a *reliability index*. This is turn is a "universal" indicator on the adequacy of a structure, and can be used as a metric to 1) assess the health of a structure, and 2) compare different structures targeted for possible remediation.

28.2 Elements of Statistics

4 Elementary statistics formulaes will be reviewed, as they are needed to properly understand structural reliability.

5 When a set of N values x_i is clustered around a particular one, then it may be useful to characterize the set by a few numbers that are related to its *moments* (the sums of integer powers of the values):

Mean: estimates the value around which the data clusters.

$$\boxed{\mu = \frac{1}{N}\sum_{i=1}^{N} x_i} \tag{28.1}$$

Skewness: characterizes the degree of asymmetry of a distribution around its mean. It is defined in a non-dimensional value. A positive one signifies a distribution with an asymmetric tail extending out toward more positive x

$$\text{Skew} = \frac{1}{N}\Sigma_{i=1}^{N}\left[\frac{x_i-\mu}{\sigma}\right]^3 \tag{28.8}$$

Kurtosis: is a nondimensional quantity which measures the "flatness" or "peakedness" of a distribution. It is normalized with respect to the curvature of a normal distribution. Hence a negative value would result from a distribution resembling a loaf of bread, while a positive one would be induced by a sharp peak:

$$\text{Kurt} = \frac{1}{N}\Sigma_{i=1}^{N}\left[\frac{x_i-\mu}{\sigma}\right]^4 - 3 \tag{28.9}$$

the -3 term makes the value zero for a normal distribution.

6 The expected value (or mean), standard deviation and coefficient of variation are interdependent: knowing any two, we can determine the third.

28.3 Distributions of Random Variables

7 Distribution of variables can be mathematically represented.

28.3.1 Uniform Distribution

8 Uniform distribution implies that any value between x_{min} and x_{max} is equaly likely to occur.

28.3.2 Normal Distribution

9 The general normal (or Gauss) distribution is given by, Fig. 28.1:

$$\boxed{\phi(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}} \tag{28.10}$$

10 A normal distribution $N(\mu,\sigma^2)$ can be normalized by defining

$$\boxed{y = \frac{x-\mu}{\sigma}} \tag{28.11}$$

and y would have a distribution $N(0,1)$:

$$\boxed{\phi(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}} \tag{28.12}$$

11 The normal distribution has been found to be an excellent approximation to a large class of distributions, and has some very desirable mathematical properties:

28.3.3 Lognormal Distribution

12 A random variable is lognormally distributed if the natural logarithm of the variable is normally distributed.

13 It provides a reasonable shape when the coefficient of variation is large.

28.3.4 Beta Distribution

14 Beta distributions are very flexible and can assume a variety of shapes including the normal and uniform distributions as special cases.

15 On the other hand, the beta distribution requires four parameters.

16 Beta distributions are selected when a particular shape for the probability density function is desired.

28.3.5 BiNormal distribution

28.4 Reliability Index

28.4.1 Performance Function Identification

17 Designating F the capacity to demand ratio C/D (or *performance function*), in general F is a function of one or more variables x_i which describe the geometry, material, loads, and boundary conditions

$$F = F(x_i) \tag{28.17}$$

and thus F is in turn a random variable with its own probability distribution function, Fig. 28.2.

18 A performance function evaluation typically require a structural analysis, this may range from a simple calculation to a detailled finite element study.

28.4.2 Definitions

19 Reliability indices, β are used as a relative measure of the reliability or confidence in the ability of a structure to perform its function in a satisfactory manner. In other words they are a measure of the performance function.

20 Probabilistic methods are used to systematically evaluate uncertainties in parameters that affect structural performance, and there is a relation between the reliability index and risk.

21 Reliability index is defined in terms of the performance function capacity C, and the applied load or demand D. It is assumed that both C and D are **random variables**.

22 The **safety margin** is defined as $Y = C - D$. Failure would occur if $Y < 0$ Next, C and D can be combined and the result expressed logarithmically, Fig. 28.2.

$$X = \ln \frac{C}{D} \tag{28.18}$$

30 The objective is to determine the mean and standard deviation of the performance function defined in terms of C/D.

31 Those two parameters, in turn, will later be required to compute the reliability index.

28.4.3.1 Direct Integration

32 Given a function random variable x, the mean value of the function is obtained by integrating the function over the probability distribution function of the random variable

$$\mu[F(x)] = \int_{-\infty}^{\infty} g(x) f(x) dx \tag{28.21}$$

33 For more than one variable,

$$\mu[F(x)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(x_1, x_2, \cdots, x_n) F(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n \tag{28.22}$$

34 Note that in practice, the function $F(x)$ is very rarely available for practical problems, and hence this method is seldom used.

28.4.3.2 Monte Carlo Simulation

35 The performance function is evaluated for many possible values of the random variables.

36 Assuming that all variables have a normal distribution, then this is done through the following algorithm

1. initialize random number generators
2. Perform n analysis, for each one:
   (a) For each variable, determine a random number for the given distribution
   (b) Transform the random number
   (c) Analyse
   (d) Determine the performance function, and store the results
3. From all the analyses, determine the mean and the standard deviation, compute the reliability index.
4. Count the number of analyses, n_f which performance function indicate failure, the likelihood of structural failure will be $p(f) = n_f/n$.

37 A sample program (all subroutines are taken from (Press, Flannery, Teukolvsky and Vetterling 1988) which generates n normally distributed data points, and then analyze the results, determines mean and standard deviation, and sort them (for histogram plotting), is shown below:

```
program nice
parameter(ns=100000)
real x(ns), mean, sd
write(*,*)'enter mean, standard-deviation and n '
read(*,*)mean,sd,n
```

```
      ix3=mod(ia3*ix3+ic3,m3)
      j=1+(97*ix3)/m3
      if(j.gt.97.or.j.lt.1)pause
      ran1=r(j)
      r(j)=(float(ix1)+float(ix2)*rm2)*rm1
      return
      end
c-----------------------------------------------------------------------------
      subroutine moment(data,n,ave,adev,sdev,var,skew,curt)
c given array of data of length N, returns the mean AVE, average
c      deviation ADEV, standard deviation SDEV, variance VAR, skewness SKEW,
c      and kurtosis CURT
      dimension data(n)
      if(n.le.1)pause 'n must be at least 2'
      s=0.
      do 11 j=1,n
        s=s+data(j)
11    continue
      ave=s/n
      adev=0.
      var=0.
      skew=0.
      curt=0.
      do 12 j=1,n
        s=data(j)-ave
        adev=adev+abs(s)
        p=s*s
        var=var+p
        p=p*s
        skew=skew+p
        p=p*s
        curt=curt+p
12    continue
      adev=adev/n
      var=var/(n-1)
      sdev=sqrt(var)
      if(var.ne.0.)then
        skew=skew/(n*sdev**3)
        curt=curt/(n*var**2)-3.
      else
        pause 'no skew or kurtosis when zero variance'
      endif
      return
      end
c------------------------------------------------------------------------------
      subroutine sort(n,ra)
      dimension ra(n)
      l=n/2+1
      ir=n
10    continue
        if(l.gt.1)then
          l=l-1
          rra=ra(l)
        else
          rra=ra(ir)
          ra(ir)=ra(1)
          ir=ir-1
          if(ir.eq.1)then
            ra(1)=rra
```

42 This is accomplished by limiting ourselves to all possible combinations of $\mu_i \pm \sigma_i$.

43 For each analysis we determine SFF, as well as its logarithm.

44 Mean and standard deviation of the logarithmic values are then determined from the 2^n analyses, and then β is the ratio of the mean to the standard deviation.

28.4.3.4 Taylor's Series-Finite Difference Estimation

45 In the previous method, we have cut down the number of deterministic analyses to 2^n, in the following method, we reduce it even further to $2n+1$, (US Army Corps of Engineers 1992, US Army Corps of Engineers 1993, Bryant, Brokaw and Mlakar 1993).

46 This simplified approach starts with the first order Taylor series expansion of Eq. 28.17 about the mean and limited to linear terms, (Benjamin and Cornell 1970).

$$\boxed{\mu_F = F(\mu_i)} \tag{28.23}$$

where μ_i is the mean for all random variables.

47 For independent random variables, the variance can be approximated by

$$\begin{aligned} Var(F) = \sigma_F^2 &= \sum \left(\frac{\partial F}{\partial x_i} \sigma_i \right)^2 && \text{(28.24-a)} \\ \frac{\partial F}{\partial x_i} &\approx \frac{F_i^+ - F_i^-}{2\sigma_i} && \text{(28.24-b)} \\ F_i^+ &= F(\mu_1, \cdots, \mu_i + \sigma_i, \cdots, \mu_n) && \text{(28.24-c)} \\ F_i^- &= F(\mu_1, \cdots, \mu_i - \sigma_i, \cdots, \mu_n) && \text{(28.24-d)} \end{aligned}$$

where σ_i are the standard deviations of the variables. Hence,

$$\boxed{\sigma_F = \sum \left(\frac{F_i^+ - F_i^-}{2} \right)} \tag{28.25}$$

Finally, the reliability index is given by

$$\boxed{\beta = \frac{\ln \mu_F}{\sigma_F}} \tag{28.26}$$

48 The procedure can be summarized as follows:

1. Perform an initial analysis in which all variables are set equal to their mean value. This analysis provides the mean μ.
2. Perform $2n$ analysis, in which all variables are set equal to their mean values, except variable i, which assumes a value equal to $\mu_i + \sigma_i$, and then $\mu_i - \sigma_i$.
3. For each pair of analysis in which variable x_i is modified, determine

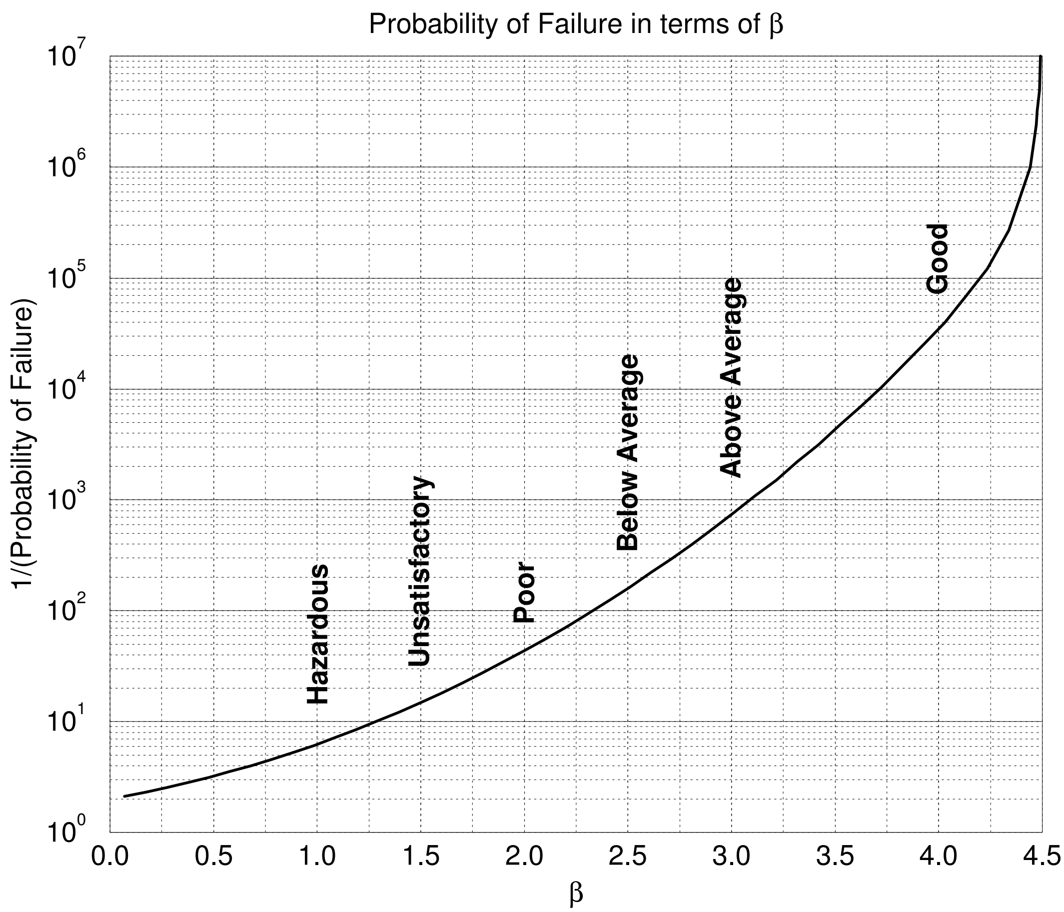

Figure 28.3: Probability of Failure in terms of β