# Chapter 17: Modelling – Vibrating String, Wave Equation

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## Introduction

One of the most important applications of partial differential equations in engineering is the modeling of physical systems involving vibrations. In structural engineering, cables, suspension bridges, and other elastic members often exhibit vibrational behavior. The classical example of such a system is the vibrating string, which is mathematically modeled using the **wave equation**, a second-order partial differential equation.

In this chapter, we derive the wave equation for a vibrating string, analyze its mathematical properties, and explore the solution techniques applicable under specific initial and boundary conditions. The understanding of wave motion and its mathematical representation is fundamental for analyzing stress, displacement, and energy transmission in civil structures.

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## 17.1 Assumptions for the Vibrating String Model

To derive the wave equation, we consider an idealized model of a vibrating string under the following assumptions:

1. The string is perfectly flexible and stretched tightly between two fixed ends.
2. The motion of the string is restricted to a single plane (e.g., vertical displacement only).
3. The tension in the string remains constant during vibration.
4. The string has uniform linear density (mass per unit length), denoted by $\rho$.
5. Damping effects (due to air resistance or internal friction) are neglected.
6. Transverse displacement is small, allowing linear approximations of slope.

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## 17.2 Derivation of the One-Dimensional Wave Equation

Let the string extend from $x = 0$ to $x = L$, and let $u(x, t)$ denote the transverse displacement at position $x$ and time $t$.

Consider a small element of the string between $x$ and $x + \Delta x$. Let the tension at position $x$ be $T$, and the angles made by the string with the horizontal at $x$ and $x + \Delta x$ be $\theta$ and $\theta + \Delta\theta$, respectively.

**Transverse Force Balance:**

Using Newton's second law in the vertical direction:

$$\text{Net vertical force} = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

$$T \sin(\theta + \Delta\theta) - T \sin(\theta) \approx T \left( \frac{\partial u}{\partial x}(x + \Delta x) - \frac{\partial u}{\partial x}(x) \right)$$

$$= T \frac{\partial^2 u}{\partial x^2} \Delta x$$

Equating force and acceleration:

$$T \frac{\partial^2 u}{\partial x^2} \Delta x = \rho \Delta x \frac{\partial^2 u}{\partial t^2}$$

Cancelling $\Delta x$, we obtain the **one-dimensional wave equation**:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $c = \sqrt{T/\rho}$ is the wave speed.

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## 17.3 Boundary and Initial Conditions

For a well-posed problem, the wave equation must be accompanied by appropriate boundary and initial conditions.

**Boundary Conditions (BC):**

For a string fixed at both ends:

$$u(0, t) = 0, \quad u(L, t) = 0 \quad \text{for all } t \geq 0$$

**Initial Conditions (IC):**

Let the initial shape and velocity of the string be given by:

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = g(x) \quad \text{for } 0 \leq x \leq L$$

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## 17.4 Method of Separation of Variables

To solve the wave equation, we use the method of **separation of variables**. Assume a solution of the form:

$$u(x,t) = X(x)T(t)$$

Substitute into the wave equation:

$$X(x)T''(t) = c^2 X''(x)T(t) \Rightarrow \frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

We now solve two ordinary differential equations:

1. **Spatial ODE** (Sturm-Liouville problem):

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0$$

2. **Temporal ODE**:

$$T''(t) + \lambda c^2 T(t) = 0$$

**Solving the Spatial ODE:**

The non-trivial solution exists only when $\lambda = \left(\frac{n\pi}{L}\right)^2$, giving:

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots$$

**Solving the Temporal ODE:**

$$T_n(t) = A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)$$

Hence, the general solution is:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

## 17.5 Determination of Coefficients Using Fourier Series

From the initial conditions:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \Rightarrow A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\frac{\partial u}{\partial t}(x,t)\bigg|_{t=0} = g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right) \Rightarrow B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

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## 17.6 Properties of the Solution

- **Superposition Principle**: The linearity of the wave equation allows superposition of solutions.
- **Wave Propagation**: Disturbances propagate along the string with speed $c$, determined by tension and linear density.
- **Standing Waves**: The natural modes of vibration correspond to standing waves with fixed nodes and antinodes.
- **Energy Conservation**: In the absence of damping, total mechanical energy is conserved.

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## 17.7 D'Alembert's Solution (Infinite String Case)

For an infinite string, the general solution to the wave equation is given by D'Alembert's formula:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

This represents two waves traveling in opposite directions with speed $c$, modified by the initial velocity profile.

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## 17.8 Applications in Civil Engineering

- Modeling vibration in suspension bridges and cables.
- Earthquake wave propagation through foundations.
- Acoustic vibration analysis in buildings.
- Designing materials to reduce vibrational amplitudes and resonance.

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## 17.9 Eigenvalues and Modes of Vibration

In structural systems, each solution of the wave equation with fixed boundary conditions corresponds to a **natural mode** or **normal mode** of vibration. These modes have critical engineering significance:

### 17.9.1 Eigenvalues $\lambda_n$

Recall from separation of variables:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Each $\lambda_n$ corresponds to a **mode shape**:

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

and associated **natural frequency**:

$$\omega_n = \frac{n\pi c}{L}$$

These frequencies must be analyzed to **avoid resonance** in structural designs.

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## 17.10 Principle of Superposition and Modal Analysis

For small vibrations, due to the linearity of the wave equation, any complex initial shape or excitation can be decomposed into a sum of natural modes.

This is especially useful in structural dynamics:

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n(x) q_n(t)$$

Where:

- $\phi_n(x)$ are spatial mode shapes (eigenfunctions),
- $q_n(t)$ are time-dependent modal amplitudes.

**Use in Civil Engineering:**

- Modal analysis allows engineers to compute **displacements, stresses, and accelerations** due to dynamic loading (e.g., wind or seismic).
- Helps optimize damping strategies to suppress harmful vibrations.

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## 17.11 Two-Dimensional Wave Equation

While the one-dimensional string is a foundational case, real-world structures often require higher-dimensional models.

The **2D wave equation** (for membranes, plates, etc.) is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

**Application in Civil Engineering:**

- Models **vibration of rectangular plates** (floors, steel plates, bridge decks).
- Requires boundary conditions on all four sides and solution using **double Fourier series** or **numerical methods** like Finite Element Method (FEM).

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## 17.12 Energy in a Vibrating String

Let us define the energy in the vibrating string, which is conserved in the absence of damping.

**Kinetic Energy $E_k$:**

$$E_k(t) = \frac{1}{2}\rho \int_0^L \left( \frac{\partial u}{\partial t} \right)^2 dx$$

**Potential Energy $E_p$:**

$$E_p(t) = \frac{1}{2}T \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 dx$$

**Total Mechanical Energy:**

$$E(t) = E_k(t) + E_p(t) = \text{constant in time}$$

This conservation law is crucial in dynamic structural analysis and ensures the correctness of numerical simulations.

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## 17.13 Effect of Damping

In real engineering systems, damping is **non-negligible**. Including damping modifies the wave equation:

$$\frac{\partial^2 u}{\partial t^2} + 2\beta \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Where $\beta$ is the damping coefficient.

- **Underdamped Systems**: Oscillate and gradually decay.
- **Critically Damped**: Return to rest as quickly as possible without oscillating.
- **Overdamped**: Return to rest slowly without oscillating.

**Engineering Use:**

Damping helps **control vibrations** and **improve structural comfort and safety** (e.g., damping devices in tall buildings and bridges).

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## 17.14 Numerical Solution Techniques

Analytical methods are often impractical for complex geometries or boundary conditions. Numerical techniques are vital.

### 17.14.1 Finite Difference Method (FDM)

Discretizes both space and time:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \quad \frac{\partial^2 u}{\partial t^2} \approx \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2}$$

Used to build iterative schemes to approximate $u(x, t)$.

### 17.14.2 Finite Element Method (FEM)

- Approximates solution using piecewise polynomials (basis functions).
- Efficient for complex structures and boundary conditions.
- Standard in commercial civil engineering software (e.g., ANSYS, ABAQUS).

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## 17.15 Real-World Examples and Case Studies

**1. Bridge Cables (e.g., Howrah Bridge, Golden Gate)**

- Cables behave like vibrating strings.
- Tension and length determine wave speed.
- Used in **wind load and resonance analysis**.

**2. Tall Buildings and Earthquakes**

- Buildings are modeled as vibrating beams or strings with distributed mass.
- Wave equations help simulate how **seismic energy travels vertically** through the structure.

**3. Railway Tracks**

- Can be modeled as semi-infinite strings on elastic foundations.
- Vibration control is essential for **comfort and structural integrity**.

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## 17.16 Extension to Nonlinear Wave Equations

In real materials, at high amplitudes or complex materials, **nonlinear effects** appear:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + \alpha u^3$$

- Appears in large amplitude oscillations or nonlinear elastic materials.
- Solved using **perturbation methods**, **numerical schemes**, or **variational techniques**.

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## 17.17 Challenges in Structural Vibration Analysis

- Accurate material modeling (Young's modulus, damping).
- Multi-physics coupling (e.g., wind + structure + foundation).
- Time-varying loads (seismic excitation, moving vehicles).
- Sensitivity to boundary and initial conditions.
- High computational cost for large structures.

These challenges make **wave equation modeling a cornerstone** of advanced structural dynamics and safety analysis in Civil Engineering.

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