**Discrete Mathematics**
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**Lecture -46**
**Proof of Hall's Marriage Theorem**

Hello everyone, welcome to this lecture. The plan for this lecture is as follows.

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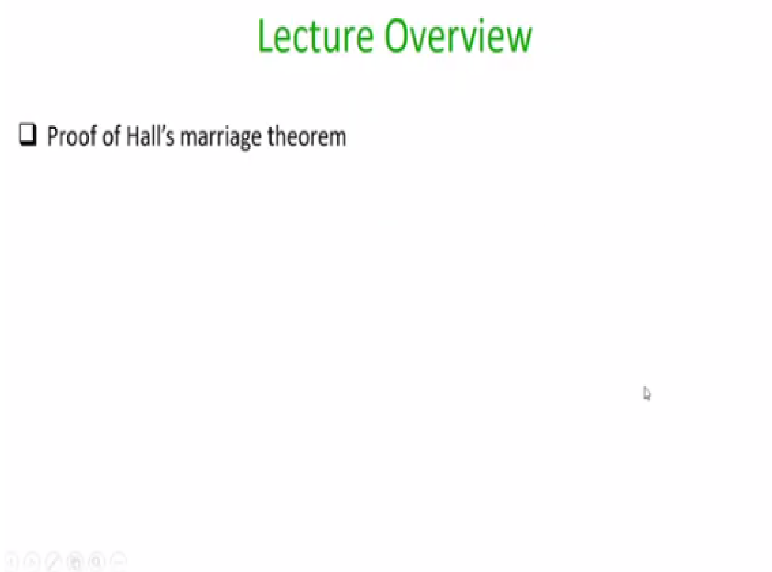


In this lecture we will see the proof of Hall's Marriage Theorem that we have discussed in the last lecture.

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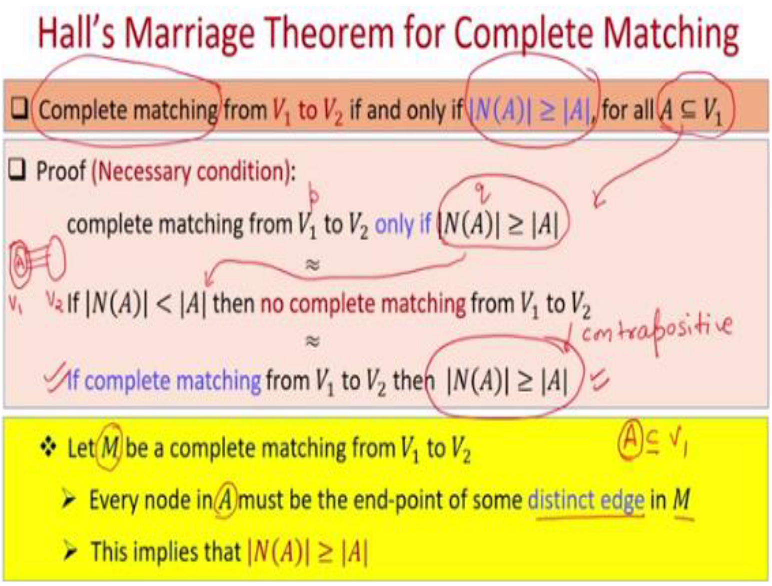

So, just to recap the theorem statement of Hall's Marriage Theorem is the following. It says that if you have a bipartite graph with bi partition ($V_1$, $V_2$) and if you want to find out whether there exists a complete matching from the subset $V_1$ to subset $V_2$ then it is possible if and only if $|N(A)| \geq |A|$,for any subset $A \subseteq V1$. So, this condition is both necessary as well as sufficient.

So let us first prove the necessary condition that indeed this condition is necessary for the existence of a complete matching. So, what exactly we want to prove here? We want to prove that complete matching from $V_1$ to $V_2$ is possible only if this condition is true. Of course, this means this condition has to be true $\forall A \subseteq V1$. So, that is implicit here. So, recall that the way we can interpret an only if statement is the following. Now if this is the p part and if this is the q part, then the way to interpret this only if condition is that if the condition after only if it is not there then whatever is there before only if that would not happen.

So, the condition that q does not happen means there exist at the least some $A \subseteq V_1$ such that the number of neighbours of that subset A is less than the number of nodes. If that is the case then we have to argue that no complete matching is possible from the vertex set $V_1$ to the vertex set $V_2$. That is what we want to prove. And the contrapositive of the statement is the following: the contrapositive says that if complete matching from the vertex set $V_1 \rightarrow V_2$ is there then you take any $A \subseteq V_1$, the number of neighbours of that subset A should be at least as large as the number of nodes in the subset for any $A \subseteq V_1$. So, that is what we want to prove here. So, this is the final thing we will prove by proving the necessary condition so and we will give a direct proof. We do not need any fancy thing here.

So, imagine there is a complete matching from $V_1 \rightarrow V_2$ and let that complete matching be denoted by M. So, if that is the case, we have to show that you take any $A \subseteq V_1$, this condition holds that is what we have to show. So, now let us focus on the nodes in A. So, remember we are considering the following you have the bipartition ($V_1$, $V_2$) and you have a subset A and you also have a complete matching M, match with respect to which all the vertices in $V_1$ are matched.

That also means that all the vertices in the subset A are also matched with respect to the same matching M. Because $A \subseteq V_1$, so that means every node in A must be the end point of some distinct edge in the complete matching that you have the found from $V_1$ to $V_2$ and that is possible only if the number of neighbours of the subset A is as large as the number of nodes in A.

Because if at all you are able to find out if you are able to match all the vertices in A using the collection of edges in M and as per the definition of matching, two distinct edges have distinct end points so that automatically means that this condition is true. So, that is the proof of the necessary condition.

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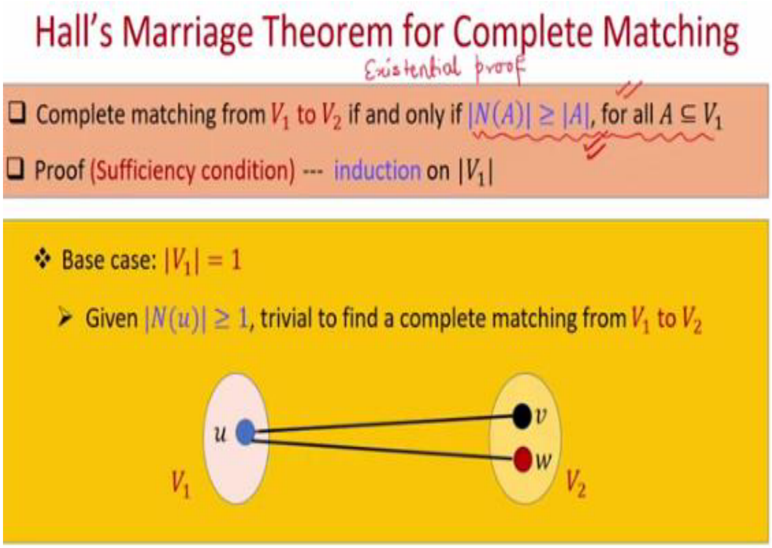


Now let us prove the sufficiency condition that means we want to prove that if this condition is ensured that means if you have a bipartition ($V_1$, $V_2$) and if it is ensured that you take any $A \subseteq V_1$ the number of neighbours, $N(A) \geq |A|$, that is guaranteed then we have to show that there exists a complete matching from $V_1$ to $V_2$. And we will give an existential proof here.

What do we mean by existential proof? We will show that if this condition is ensured then there exist at least one complete matching from the vertex set $V_1$ to $V_2$ and that existential proof will be given by induction on the cardinality of your vertex set $V_1$, $|V|$. So, we will first prove the base case. So, assume that you have a bipartite graph with bi partition ($V_1$, $V_2$) and where there is only one vertex in $V_1$ and all other vertices of your graph are in the subset $V_2$ and this condition is ensured for your ($V_1$, $V_2$). If that is the case since my vertex set $V_1$ has only one node call it u. The only subset A possible for $V_1$ is the subset $V_1$ itself. Of course, we can have the empty subset A of $V_1$ but that is not interesting. We take $A \subseteq V_1$ and $A \neq \phi$.

That is possible here is the subset $V_1$ itself A being the $V_1$ itself and since this condition is guaranteed that means there is at least one node the node u has at least one neighbour in $V_2$. It may have more than one neighbour as well that is also possible but since $N(A) \geq |A|$ and if I take $A = V_1$ , the base case ensures that the node u has at least one neighbour in the subset $V_2$.

And if that is the case then it is very trivial to find out the complete matching from $V_1$ to $V_2$. The complete matching will be, just take one of the edges with u as the one of the end points and that will be a complete matching from $V_1$ to $V_2$.

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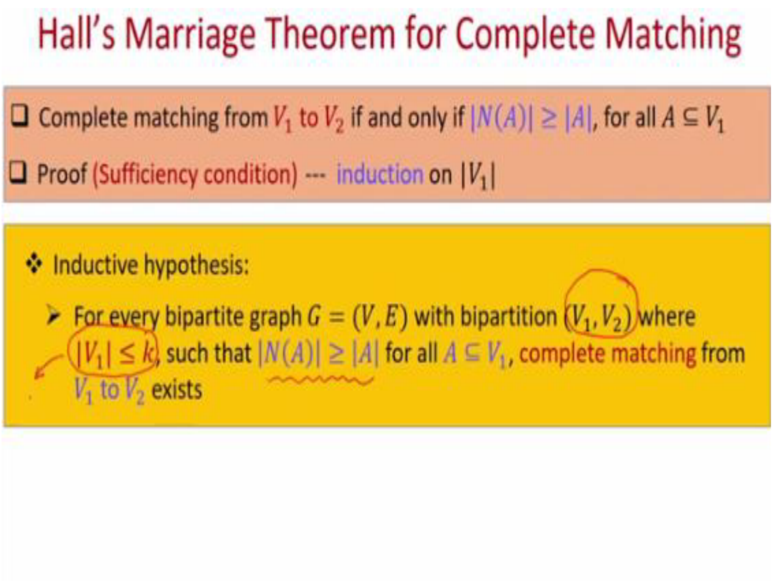


So, now let us go to the inductive step and for the inductive step we first assume the inductive hypothesis. So, my inductive hypothesis is the following. I assume here that you take any bipartite graph with bipartition ($V_1$, $V_2$) such that $|V_1| \leq k$ and if it is ensured that for any $A \subseteq V_1$, the number of neighbours of A is at least as large as the number of nodes in A, then a complete matching is there from $V_1$ to $V_2$. That is my inductive hypothesis. I am assuming this to be true for all bipartite graphs where $|V_1| \leq k$ .

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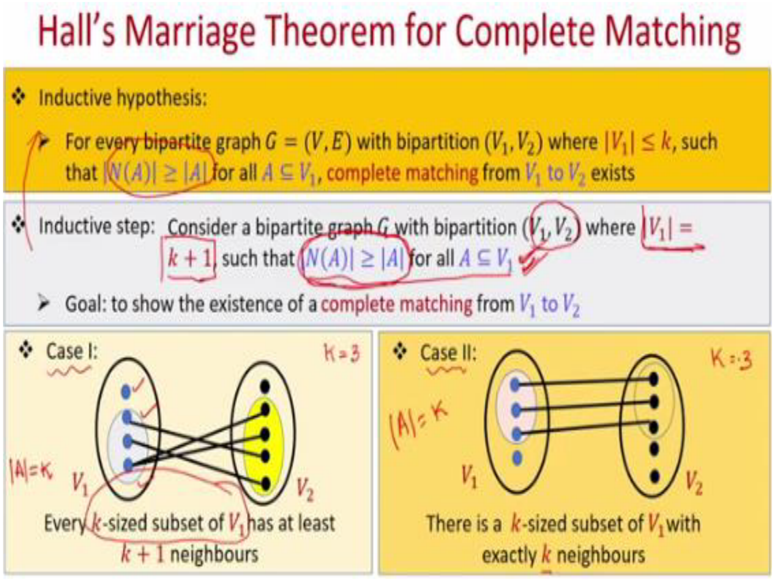


Now I have to go to the inductive step and I have to show that assuming the base case and assuming the inductive hypothesis to be true, I have to prove that the statement or the sufficiency condition is true even for a bipartite graph where $|V_1| = k + 1$ provided this condition is ensured in that graph. So, I consider an arbitrary bipartite graph G with bipartition ($V_1$, $V_2$).

And the cardinality of $|V_1| = k + 1$ such that it is ensured that for any $A \subseteq V_1$, the number of neighbours of the subset A is as large as the number of nodes in A that is given to me. My goal is to show the existence of a complete matching in my graph G from the vertex set $V_1$ to $V_2$ that means I have to give you a matching I have to show that there exists a matching with respect to which all the vertices of the subset $V_1$ will be matched.

And I have to use the inductive hypothesis because right now I am considering the case when my cardinality of $V_1$ is k + 1. So, as a principle of inductive proof we have to somehow reduce a graph, a bipartite graph where $V_1$ is of cardinality k + 1 to another bipartite graph where the bipartition has the property that the corresponding $V_1$ has cardinality k. And then I have to use the inductive hypothesis on that graph and show the existence of a complete matching in that reduced graph. And based on the complete matching that I have in the reduced graph I have to show that I can build upon that complete matching in the reduced graph and give you a complete matching for the bigger graph G. So, that will be the proof strategy. So, for doing that what I am going to do is I am going to exploit this condition.

So, I am assuming here that my graph G is such that for any subset A of the set $V_1$ the number of neighbours of A is as large is at least as large as the number of nodes in A. So, now there could be two possible cases here. Case 1 is the following, your graph G is such that for every k-sized subset of $V_1$ that subset has at least k + 1 neighbours in the subset $V_2$. So, here I am focusing on the case where A is exactly equal to k.

So, my case one is you take all your graph G is such that you take any $|A| = k$ in your $V_1$ that has at least k + 1 neighbours in the subset $V_2$. So, for instance if I take k equal to say 3. So what I am saying here is your graph G is such that you take any subset of three nodes in your $V_1$, it will have 4 or more number of neighbours in $V_2$. So, for instance if you take the first 3 nodes, it will have 4 nodes, 4 neighbours in $V_2$ or if you take the last 3 nodes then also it has 4 or more neighbours in $V_2$.

Or even if you take say for instance the first node, the second node and the fourth node that also will have 4 or more number of neighbours in $V_2$ and so on. So, that is case 1, that means your graph G is such that this condition is there. And my case 2 could be the following. I have a k-sized subset of $V_1$ which has exactly k neighbours in $V_2$. So, pictorially you can imagine I am talking about the case where your graph G is such that even though this condition is true, but as part of that condition you have a subset A of k nodes in $V_1$ which has exactly k neighbours in $V_2$ that is the case that does not violate this condition. This condition is still satisfied even for that subset A because this condition says that $N(A) \geq |A|$. So, even if it is equal to the number of nodes in A that means the condition is satisfied.

So, my case 2 is talking about a possibility where in my graph G, I have a subset A of k nodes which has exactly k neighbours in the subset $V_2$. So, again for demonstration here I am taking the case of $k = 3$. So, these are the only 2 possible cases with respect to my graph G and in both the cases I have to show the existence of a complete matching from $V_1$ to $V_2$ and in both the cases I will be using the inductive hypothesis. So, let us first consider case 1.

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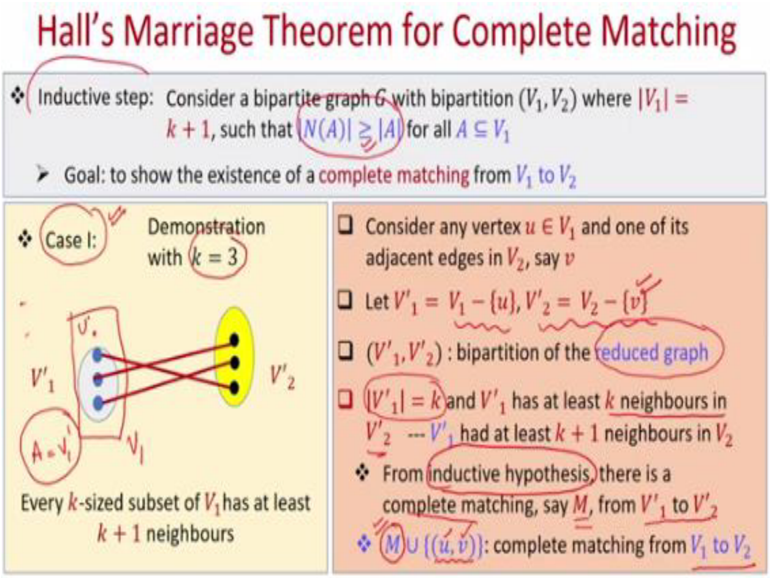


And for demonstration purpose I am taking $k = 3$. So, this is the case where my graph G is a bipartite graph with bipartition ($V_1$, $V_2$) and $|V_1| = k + 1$ and this condition is ensured in my graph G and this condition is ensured in such a way that you take every k-sized subset of $V_1$ in G, it has k + 1 or more number of neighbours in $V_2$. That is the case I am considering right now.

And my goal is to show the existence of a complete matching from $V_1$ to $V_2$. So, here is how I will find the complete matching. So, you consider any vertex u from $V_1$ you are free to use any vertex, just for simplicity I am taking the first vertex. And remember my goal is to reduce this graph G where $|V_1| = k + 1$ to another bipartite graph where the cardinality of the corresponding $V_1$ is k so that I can use the inductive hypothesis.

So, for that only I am considering an arbitrary vertex u in the subset $V_1$ and I am focusing on one of its neighbours in $V_2$. So, for instance let it be u and its corresponding neighbour v is there. By the way what is the guarantee that the node u that I have picked here has at least one neighbour v in $V_2$, well that is coming because of the base case if I consider the case where A is equal to one.

And the subset A being the set consisting of node u, then as per the condition the number of neighbours of u is one or more than one. So, that means at least one neighbour of u is there in my graph and that neighbour has to be in the subset $V_2$ because I am considering a bipartite graph. So, out of all the neighbours of u, I am just picking some arbitrary neighbour call it v and then what I do is I reduce my graph to a following graph.

I remove the node u from my graph and I remove the node v from the graph and I remove this edge because this edge now is part of my matching. Remember my goal is to find out the complete matching in the overall graph, so one of the edges of that complete matching is the edge (u, v) and that will ensure that the node u is matched. Now I have to take care of ensuring that the remaining k nodes of $V_1$ are also somehow matched.

So, because of this reduction now I will get a new graph and that new graph will also be a bipartite graph because my original graph was a bipartite graph and the only thing that I have changed is I have removed the node u, I have removed the node v and I have removed the edge between u and v and all the edges which has u as one of its end point. And all the edges which has v as this endpoint.

So, that will ensure that my new graph which I am calling as the reduced graph is still a bipartite graph and the corresponding bipartition of the reduced graph will be $V_1$', $V_2$'. So, $V_1' = V_1 - u$ and $V_2' = V_2 - v$. Now what can we say about the cardinality of $|V_1'|$? It will be k. And what can I say about the cardinality or the number of neighbours of $V_1$' that are there in $V_2$'?

My claim here is that, the nodes in $V_1$' has k or more number of neighbour in $V_2$' in my reduced graph. This is because in my original graph G not the reduced graph in my original graph G, if I take the case where $A = V_1'$ i then since I am in case 1 it would have been ensured that in my graph G, this subset A namely the subset $V_1$' has k + 1 or more number of neighbours in G, because I am in case 1. One of the neighbours of $V_1$' could be the node v which I have deleted and taken as part of the edge (u, v) in my complete matching which I am trying to build. But even if I now remove the node v from the graph G in my reduced graph it will be ensured that the number of neighbours of $V_1$' will be k or more than k.

Because if $N(V_1') = k - 1$ in my reduced graph, then I get the implication that in my bigger graph namely the original graph, the subset $V_1$' has exactly k neighbours. But that goes against the assumption that I am in case 1 and in case 1, I am assuming that each k-sized subset of $V_1$ in the graph G has k + 1 or more number of neighbours. Now if the subset $V_1$' has at least k number of neighbours in $V_2$' then I can use my inductive hypothesis.

And as per my inductive hypothesis if you have a bipartite graph where the cardinality of the first set in your bipartition is exactly k and if it is ensured that, you take any subset of $V_1$’, it has at least as many neighbours as the number of nodes in A. Then as per my inductive hypothesis, I know that there exists a complete matching in my reduced graph. I say I stress here in the reduced graph which will ensure that all the vertices in $V_1$’ are matched.

That means it will be a complete matching from $V_1$’ to $V_2$’. Now take that complete matching and to that complete matching add the edge (u, v) and that will give you now a complete matching in the original graph G matching or ensuring that all the vertices of $V_1$ are matched. So, it will be a complete matching from $V_1$ to $V_2$. And why this is a valid matching because in the matching M that you are finding in the reduced graph none of the edges will have the node u or the node v as its end point.

Because the node u and node v or none of the edges incident with u or v are present in your reduced graph. Because they were present in your original graph and you have removed the node u, node v and all the associated edges and got the reduced graph and your matching M is in the reduced graph and if in that matching you add this edge (u, v) that will ensure that your original $V_1$ which also had the node u. So, it is completely covered or it is ensured that all the nodes in $V_1$ are matched with respect to this bigger match. So, that is the proof for case 1.

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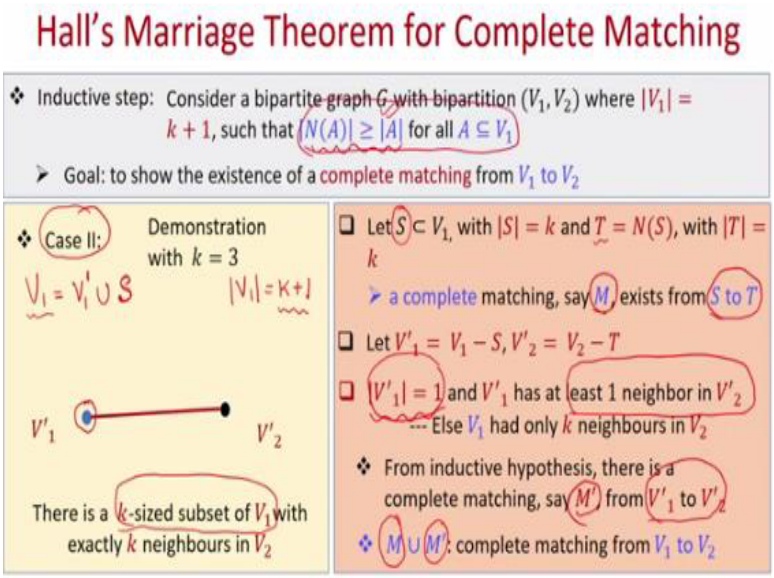


Whereas now let us focus on case 2 and the case 2 is slightly subtle here because here we are in the case where we are assuming that there is some k-sized subset of $V_1$ which has exactly k neighbours in $V_2$ and in this case, we cannot run the argument that we used for case 1. In case

1, what we did is we arbitrarily picked some node in $V_1$ and matched it by taking one of the edges incident with that node.

And argued that even if I remove u from my graph the remaining $V_1$ namely $V_1$' it will be ensured that it has k or more number of neighbours in the reduced graph but that would not happen because of this specific case. It might be the possible it might be possible that the node u is part of a k size subset of $V_1$ which has exactly k neighbours in $V_2$. So, when you are removing the, when you are removing the edge (u, v) from the graph and getting the reduced graph.

Then that k-sized subset may be will be now reduced to k - 1 size subset and now in the k - 1 size subset you may not have sufficient number of neighbours in the corresponding $V_2$' and you cannot run the and you cannot use the inductive hypothesis. So, you will get stuck here so we have to handle this case in a careful fashion and still show the guarantee the existence of a complete matching from $V_1$ to $V_2$.

So, what I do here is the following. Since there is at least one k-sized subset of $V_1$ which has exactly k neighbours in $V_2$, I focus on that subset call it is there might be multiple such subsets in $V_1$. I take any one of them so take the subset $|S| = k$ and focus on its neighbour set T, such that $|S| = |T| = k$. So, $S \subseteq V_1$ and $T \subseteq V_2$.

So, for instance this is your set S this is your set T. Now since $|S| = k$, I can use my inductive hypothesis and since the number of neighbours of S is as large as the number of nodes in S from inductive hypothesis a complete matching is there. So, call that complete matching as M. Now this is a complete matching from S to T not from $V_1$ to $V_2$. So, there is still one node left which is not yet matched.

Because that is not part of this matching M, so now my reduced graph will be the following. I remove the set of nodes from in S from $V_1$ and I remove the set of nodes in T from $V_2$ and get the corresponding $V_1$' and $V_2$'. So, $V_2$' may have more than one nodes as well but here for simplicity I am left with a graph which has one node in $V_1$' and one node in $V_2$'.

So, $|V_1'| = 1$, because my $V_1$ had k + 1 nodes and I removed a subset of k node so I am left with only 1 node and my claim is that $V_1$' has still at least 1 neighbour in your reduced $V_2$ namely in $V_2$'. If this is not the case then what it ensures the following: it ensures that in your original graph G, the set $V_1$ had exactly k neighbours and remember the set $V_1$ is nothing but this leftover node so your $V_1$ is nothing but your $V_1' \cup S$. So, my claim is that if in if this node which is left in $V_1$' it has no neighbour in $V_2$' then your original graph G the subset $V_1$ had exactly k neighbours. And $|V_1| = k + 1$ remember because S is of size k and you are left with one node in $V_1$' so overall $V_1$ had k + 1 nodes.

So, I get the implication that $V_1$ had exactly k neighbours that means there is an A where the number of neighbours of A is less than the number of nodes in A but that is violation of this condition it is guaranteed that you take any $A \subseteq V_1$ in your graph G. The number of neighbours is as large as the number of nodes in A. So, that means this will give you a false conclusion.

If $V_1$' if the single node in $V_1$' has no neighbour left in $V_2$', then that gives me an implication that in the original graph G if I take the set A to be the subset $V_1$ itself then it has only k neighbours namely less number of neighbours. But that goes against my assumption that in my graph G this condition is true for every subset A.

So, from my inductive hypothesis, I know that there is now a complete matching M prime also from $V_1$' to $V_2$' this is basically coming from the base case not from the inductive hypothesis because I can trigger the base case as my cardinality of $V_1$' is 1. So, my from my base case I know that since the number of neighbours of $V_1$' is as large as the number of nodes in $V_1$'.

And $V_1$' is of size 1, I can use the base case and argue that there is some complete matching M' which ensures that all the vertices of $V_1$' are matched or that matching M' is a complete matching from $V_1$' to $V_2$'. Now if I take the union of the matching M from the subset S to the subset T and the matching M' which is a complete matching from $V_1$' to $V_2$', that will ensure that now I have a complete matching from $V_1$ to $V_2$.

So, that proves the sufficiency of the condition even for case 2. So, it does not matter whether I am in case 1 or in case 2, in both the cases if this condition is ensured that means, you take any $A \subseteq V_1$, the number of neighbours $N(A)$ is as large as the number of nodes in A then there always exist a complete matching from the subset $V_1$ to subset $V_2$.

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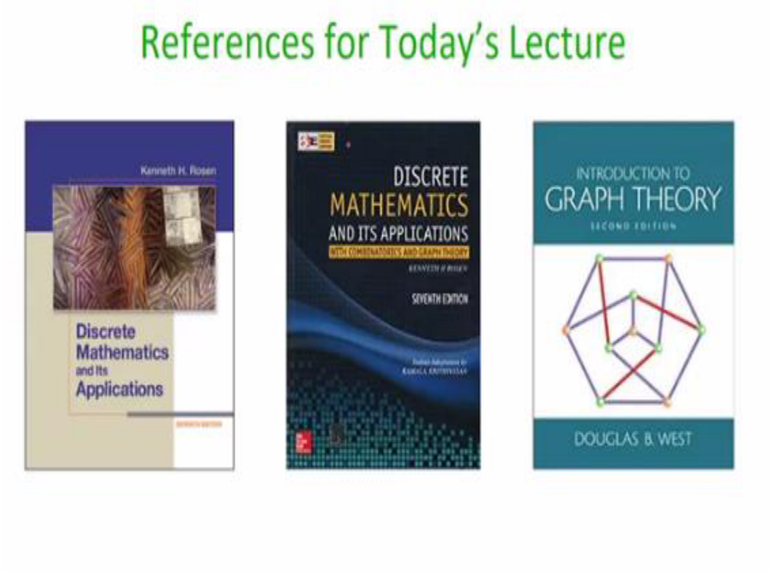


So, that brings me to the end of this lecture. These are the references for today's lecture. To summarize, in this lecture we discussed the proof of Hall's Marriage Theorem. We showed the necessary proof of this we prove the necessity condition as well as we give an existential proof for the sufficiency condition, thank you!