Chapter 6: Non-Homogeneous Equations

Introduction

In the study of differential equations, particularly in engineering applications, many physical systems are governed by **non-homogeneous differential equations**. These equations appear when external forces or inputs act on a system, such as loads on a beam, heat sources in a medium, or electrical inputs in circuits. Unlike homogeneous equations, which describe the system's natural response, **non-homogeneous differential equations describe the total response**, which includes both the natural and the forced responses.

For civil engineers, solving non-homogeneous differential equations is essential in modeling real-world scenarios like structural deflection, fluid flow under external pressure, and heat conduction with a source term. This chapter introduces the general theory and two principal methods used to solve such equations: the **method of undetermined coefficients** and the **method of variation of parameters**.

6.1 General Form of a Linear Non-Homogeneous Differential Equation

A **second-order linear non-homogeneous differential equation** with constant coefficients is given by:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Where:

- a, b, c are constants (with $a \neq 0$),
- $y(x)$ is the unknown function,
- $f(x)$ is a known function (non-zero), called the **non-homogeneous term** or **forcing function**.

General Solution Structure

The general solution $y(x)$ of the non-homogeneous equation is given by:

$$y(x) = y_h(x) + y_p(x)$$

Where:

- $y_h(x)$ is the **complementary function (CF)**: the general solution of the corresponding homogeneous equation:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

- $y_p(x)$ is the **particular integral (PI)**: a specific solution to the non-homogeneous equation.

6.2 Solving the Homogeneous Part

To find $y_h(x)$, solve the **auxiliary (or characteristic) equation**:

$$ar^2 + br + c = 0$$

Let the roots be:

- Distinct real: $r_1, r_2 \rightarrow y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
- Repeated real: $r \rightarrow y_h = (C_1 + C_2 x)e^{rx}$
- Complex: $\alpha \pm \beta i \rightarrow y_h = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$

This step is **identical** for both homogeneous and non-homogeneous equations.

6.3 Finding the Particular Integral

There are multiple methods for finding $y_p(x)$. The most common ones used in engineering applications are:

6.3.1 Method of Undetermined Coefficients

When to Use

- Only when $f(x)$ is a linear combination of functions like:
  - Polynomials (e.g. x, x^2)
  - Exponentials (e.g. e^{ax})
  - Trigonometric functions (e.g. $\sin x, \cos x$)
  - Their products (e.g. $xe^x, e^x \cos x$, etc.)

Procedure

1. **Guess a form** for $y_p(x)$, based on the form of $f(x)$, with unknown coefficients.
2. **Substitute** this guess into the differential equation.
3. **Determine the coefficients** by matching both sides.

Important Note – Modification Rule If the guessed form for y_p is a solution of the homogeneous part (i.e., appears in y_h), multiply by x or a higher power of x until linear independence is achieved.

Examples **Example 1:** Solve:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$$

Solution:

- Auxiliary Equation: $r^2 - 3r + 2 = 0 \Rightarrow r = 1, 2$
- $y_h = C_1e^x + C_2e^{2x}$

Since e^x already appears in y_h, guess $y_p = Axe^x$

- Differentiate and substitute into the original equation.
- Find A, and then write the complete solution:

$$y = C_1e^x + C_2e^{2x} + Axe^x$$

6.3.2 Method of Variation of Parameters

When to Use

- When $f(x)$ is **not** of the standard type or not suitable for undetermined coefficients.
- Works for **any** form of $f(x)$, but involves integration.

Procedure Given:

$$ay'' + by' + cy = f(x)$$

1. Solve the homogeneous equation to find $y_1(x), y_2(x)$, the two linearly independent solutions.
2. Assume:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

3. Find $u_1(x), u_2(x)$ by solving the system:

$$u_{1'}(x)y_1(x) + u_{2'}(x)y_2(x) = 0$$
$$u_{1'}(x)y_{1'}(x) + u_{2'}(x)y_{2'}(x) = \frac{f(x)}{a}$$

4. Integrate $u_{1'}(x), u_{2'}(x)$ to get u_1, u_2, then substitute into $y_p(x)$.

Example Solve:

$$y'' + y = \tan x$$

Solution:

- Homogeneous solution: $y_h = C_1 \cos x + C_2 \sin x$
- Let $y_1 = \cos x, y_2 = \sin x$

Using variation of parameters:

$$u_{1'}(x) \cos x + u_{2'}(x) \sin x = 0$$
$$-u_{1'}(x) \sin x + u_{2'}(x) \cos x = \tan x$$

Solve these equations to find $u_{1'}, u_{2'}$, integrate, and get the particular integral.

6.4 Applications in Civil Engineering

Non-homogeneous equations are vital in:

- **Beam deflection under load:** Governing differential equation:

$$EI \frac{d^4 y}{dx^4} = w(x)$$

- where $w(x)$ is the distributed load (forcing function).
- **Thermal conduction with sources:**

$$\frac{d^2 T}{dx^2} = -\frac{q(x)}{k}$$

- where $q(x)$ is the heat source.
- **Fluid flow problems** where external forces like pressure or gravity act on the system.

Being able to solve non-homogeneous equations equips civil engineers to model and analyze these physical systems accurately.

6.5 Higher-Order Non-Homogeneous Equations

In civil engineering applications, sometimes **third-order** or **fourth-order** non-homogeneous equations occur, especially in beam theory and vibration analysis.

A general **nth-order linear non-homogeneous differential equation**:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

Solution Methodology

- **Step 1: Find the Complementary Function (CF)** by solving the homogeneous equation.
- **Step 2: Use the method of undetermined coefficients or variation of parameters** to find the particular integral (PI).
- **Step 3: Combine both for the general solution:**

$$y(x) = y_h(x) + y_p(x)$$

6.6 Special Case: Resonance

In **mechanical and structural systems**, resonance occurs when the frequency of the forcing function matches the natural frequency of the system.

Example of Resonance

Consider:

$$\frac{d^2 y}{dx^2} + \omega^2 y = \cos(\omega x)$$

Here, the forcing function $\cos(\omega x)$ matches the natural frequency of the system.

Solution Approach:

- CF: $y_h = C_1 \cos(\omega x) + C_2 \sin(\omega x)$
- Since $\cos(\omega x)$ appears in the CF, guessing PI as $A \cos(\omega x) + B \sin(\omega x)$ fails.

Modification Rule: Multiply the guess by $x \rightarrow$ Try:

$$y_p = x(A \cos(\omega x) + B \sin(\omega x))$$

This reflects **resonant behavior** where the response grows with x, showing **amplification** — a critical concept in structural dynamics.

6.7 Non-Homogeneous Systems of Differential Equations

Civil engineering models often involve **multiple dependent variables** interacting. For instance:

$$\frac{dx}{dt} = 3x + 4y + \sin t$$
$$\frac{dy}{dt} = -4x + 3y + e^t$$

This system is **non-homogeneous** due to the sine and exponential terms.

Solution Outline

- Solve the **homogeneous system** using eigenvalues/eigenvectors.
- Use variation of parameters or an integrating factor matrix to find the particular solution.

This topic bridges into **matrix methods and Laplace transforms**, introduced in later chapters.

6.8 Worked Examples with Engineering Applications

Example 1: Beam Under Uniform Load

Given:

$$EI\frac{d^4y}{dx^4} = q_0$$

Where:

- EI = flexural rigidity of the beam
- q_0 = uniform load per unit length

Solution:

Rewriting:

$$\frac{d^4y}{dx^4} = \frac{q_0}{EI}$$

Integrating four times:

$$
\begin{aligned}
\frac{d^3y}{dx^3} &= \frac{q_0}{EI}x + C_1 \\
\frac{d^2y}{dx^2} &= \frac{q_0}{2EI}x^2 + C_1x + C_2 \\
\frac{dy}{dx} &= \frac{q_0}{6EI}x^3 + \frac{C_1}{2}x^2 + C_2x + C_3 \\
y(x) &= \frac{q_0}{24EI}x^4 + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3x + C_4
\end{aligned}
$$

Boundary conditions (e.g., at fixed ends) are used to determine C_1 to C_4.

Example 2: Vibration of a Damped System with Forcing

Given:

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F_0\cos(\omega t)$$

This is a **non-homogeneous second-order ODE** representing a damped forced vibration.

- Solve the homogeneous equation:

$$mr^2 + cr + k = 0$$

- Based on discriminant, determine y_h
- Guess PI using undetermined coefficients (if $\omega \neq \omega_0$) or resonance form (if $\omega = \omega_0$)

This equation is foundational in **seismic design**, **machine foundation modeling**, and **vibration control** in structures.

6.9 Conceptual Notes

- Non-homogeneous differential equations model **forced systems** — very common in civil structures where loads, vibrations, or heat sources exist.
- If the **forcing function** $f(x)$ is zero → the equation becomes **homogeneous**, representing free vibration or natural behavior.
- The **method of undetermined coefficients** is easier but limited.
- The **method of variation of parameters** is powerful and general, but involves more calculation.

6.10 Visualizing Solutions

In practical civil engineering design:

- **CF** y_h describes how the system behaves **naturally** (without external forces).
- **PI** y_p represents how the system responds **due to external forces**.
- The total solution helps predict **maximum deflection, critical points, and response time**.

Plotting these curves helps engineers verify stability, durability, and compliance with codes.
