# Chapter 8

# DEFLECTION of STRUCTRES; Geometric Methods

1 Deflections of structures must be determined in order to satisfy serviceability requirements i.e. limit deflections under *service* loads to acceptable values (such as $\frac{\Delta}{L} \leq 360$).

2 Later on, we will see that deflection calculations play an important role in the analysis of statically indeterminate structures.

3 We shall focus on flexural deformation, however the end of this chapter will review axial and torsional deformations as well.

4 Most of this chapter will be a *review* of subjects covered in *Strength of Materials*.

5 This chapter will examine deflections of structures based on geometric considerations. Later on, we will present a more pwerful method based on energy considerations.

## 8.1 Flexural Deformation

### 8.1.1 Curvature Equation

6 Let us consider a segment (between point 1 and point 2), Fig. 8.1 of a beam subjected to flexural loading.

7 The **slope** is denoted by $\theta$, the change in slope per unit length is the **curvature** $\kappa$, the **radius of curvature** is $\rho$.

8 From *Strength of Materials* we have the following relations

$$ds = \rho d\theta \Rightarrow \frac{d\theta}{ds} = \frac{1}{\rho} \tag{8.1}$$

9 We also note by extension that $\Delta s = \rho \Delta\theta$

10 As a first order approximation, and with $ds \approx dx$ and $\frac{dy}{dx} = \theta$ Eq. 8.1 becomes

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2} \tag{8.2}$$

14 Thus the slope $\theta$, curvature $\kappa$, radius of curvature $\rho$ are related to the $y$ displacement at a point $x$ along a flexural member by

$$\boxed{\kappa = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}} \tag{8.9}$$

15 If the displacements are very small, we will have $\frac{dy}{dx} << 1$, thus Eq. 8.9 reduces to

$$\boxed{\kappa = \frac{d^2y}{dx^2} = \frac{1}{\rho}} \tag{8.10}$$

### 8.1.2 Differential Equation of the Elastic Curve

16 Again with reference to Figure 8.1 a positive $d\theta$ at a positive $y$ (upper fibers) will cause a *shortening* of the upper fibers

$$\Delta u = -y\Delta\theta \tag{8.11}$$

17 This equation can be rewritten as

$$\lim_{\Delta s \to 0} \frac{\Delta u}{\Delta s} = -y \lim_{\Delta s \to 0} \frac{\Delta \theta}{\Delta s} \tag{8.12}$$

and since $\Delta s \approx \Delta x$

$$\underbrace{\frac{du}{dx}}_{\varepsilon} = -y\frac{d\theta}{dx} \tag{8.13}$$

Combining this with Eq. 8.10

$$\boxed{\frac{1}{\rho} = \kappa = -\frac{\varepsilon}{y}} \tag{8.14}$$

This is the fundamental relationship between curvature ($\kappa$), elastic curve ($y$), and linear strain ($\varepsilon$).

18 Note that so far we made no assumptions about material properties, i.e. it can be elastic or inelastic.

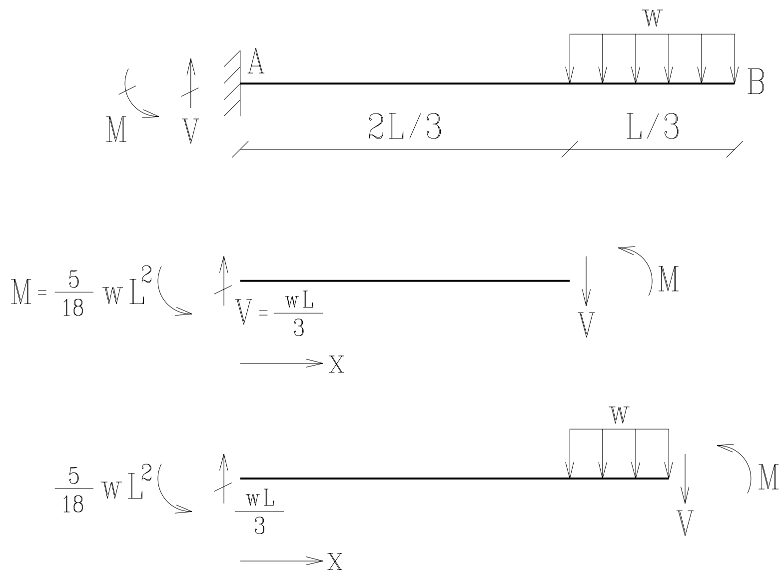
19 For the elastic case:

$$\left.\begin{array}{rcl} \varepsilon_x & = & \frac{\sigma}{E} \\ \sigma & = & -\frac{My}{I} \end{array}\right\} \varepsilon = -\frac{My}{EI} \tag{8.15}$$

Combining this last equation with Eq. 8.14 yields

$$\boxed{\frac{1}{\rho} = \frac{d\theta}{dx} = \frac{d^2y}{dx^2} = \frac{M}{EI}} \tag{8.16}$$

This fundamental equation relates moment to curvature.




**Solution:**

**At:** $0 \leq x \leq \frac{2L}{3}$

1. Moment Equation
$$EI\frac{d^2y}{dx^2} = M_x = \frac{wL}{3}x - \frac{5}{18}wL^2 \tag{8.22}$$
2. Integrate once
$$EI\frac{dy}{dx} = \frac{wL}{6}x^2 - \frac{5}{18}wL^2x + C_1 \tag{8.23}$$
However we have at $x = 0$, $\frac{dy}{dx} = 0$, $\Rightarrow C_1 = 0$
3. Integrate twice
$$EIy = \frac{wL}{18}x^3 - \frac{5wL^2}{36}x^2 + C_2 \tag{8.24}$$
Again we have at $x = 0$, $y = 0$, $\Rightarrow C_2 = 0$

**At:** $\frac{2L}{3} \leq x \leq L$

1. Moment equation
$$EI\frac{d^2y}{dx^2} = M_x = \frac{wL}{3}x - \frac{5}{18}wL^2 - w(x - \frac{2L}{3})(\frac{x - \frac{2L}{3}}{2}) \tag{8.25}$$
2. Integrate once
$$EI\frac{dy}{dx} = \frac{wL}{6}x^2 - \frac{5}{18}wL^2x - \frac{w}{6}(x - \frac{2L}{3})^3 + C_3 \tag{8.26}$$
Applying the boundary condition at $x = \frac{2L}{3}$, we must have $\frac{dy}{dx}$ equal to the value coming from the left, $\Rightarrow C_3 = 0$

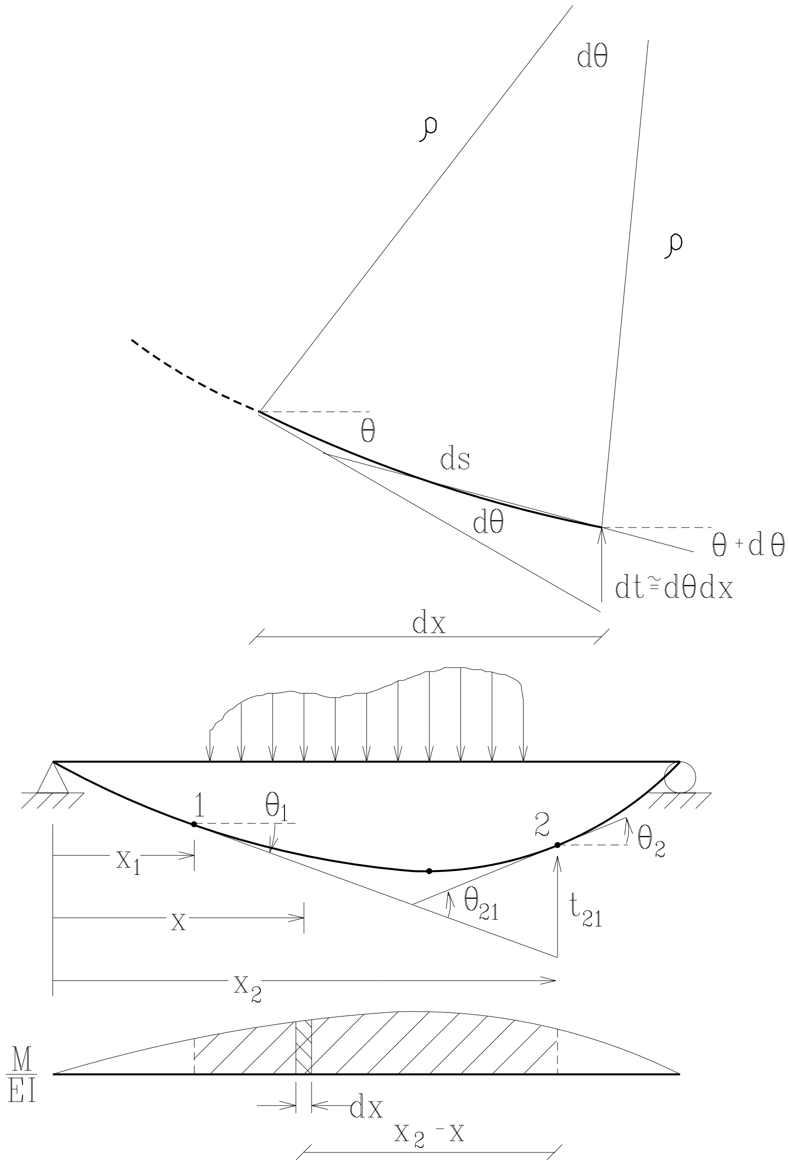



Figure 8.2: Moment Area Theorems