Chapter 19: Modelling – Membrane, Two-Dimensional Wave Equation

Introduction

Civil engineers frequently encounter problems involving vibrating surfaces—such as bridges, building floors, or membranes like drums or architectural fabrics. Understanding how these structures respond to external forces and oscillate over time is crucial for ensuring stability, safety, and performance. The behavior of such systems is modeled using **partial differential equations**, particularly the **two-dimensional wave equation**. This chapter presents the mathematical modelling of a **vibrating membrane** and develops the **two-dimensional wave equation** governing its motion.

19.1 Physical Model of a Vibrating Membrane

A **membrane** is a thin, flexible surface stretched tightly across a frame—like a drumhead. When disturbed, it vibrates in complex patterns depending on the initial force, boundary constraints, and its tension and mass.

Let us consider:

- A rectangular membrane lying in the **xy-plane**, occupying the domain $0<x<a$, $0<y<b$.
- It is tightly stretched and fixed along the boundary.
- Let $u(x,y,t)$ represent the **vertical displacement** of the membrane at position (x,y) and time t.

Assumptions:

- The membrane is homogeneous and isotropic.
- Tension T is uniform across the surface.
- The motion is small (linearization valid).
- No external force acts during motion (free vibration).

19.2 Derivation of the Two-Dimensional Wave Equation

Let:

- ρ: mass per unit area (surface density),
- T: tension per unit length (N/m),
- $u(x, y, t)$: vertical displacement.

Small Element Analysis

Take a small rectangular element $\Delta x \times \Delta y$. The vertical force due to tension is:

$$F = T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\Delta x \Delta y$$

According to Newton's second law:

$$\rho \Delta x \Delta y \frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\Delta x \Delta y$$

Dividing both sides:

$$\rho \frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Or:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

where $c^2 = \frac{T}{\rho}$ is the square of the wave speed.

19.3 The Two-Dimensional Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

This is the **two-dimensional wave equation**, a second-order linear PDE describing wave motion in a rectangular membrane.

19.4 Boundary and Initial Conditions

Boundary Conditions (Dirichlet)

Since the membrane is fixed at the boundary:

$$u(0, y, t) = u(a, y, t) = 0, \forall\, 0 < y < b, t > 0$$

$$u(x, 0, t) = u(x, b, t) = 0, \forall\, 0 < x < a, t > 0$$

Initial Conditions

At $t = 0$:

$$u(x, y, 0) = f(x, y) \text{(initial shape)}$$

$$\frac{\partial u}{\partial t}(x, y, 0) = g(x, y) \text{(initial velocity)}$$

19.5 Solution by Separation of Variables

Assume:

$$u(x, y, t) = X(x) Y(y) T(t)$$

Substituting into the wave equation:

$$X Y \frac{d^2 T}{d t^2} = c^2 \left(Y T \frac{d^2 X}{d x^2} + X T \frac{d^2 Y}{d y^2} \right)$$

Dividing both sides by $X Y T$:

$$\frac{1}{T} \frac{d^2 T}{d t^2} = c^2 \left(\frac{1}{X} \frac{d^2 X}{d x^2} + \frac{1}{Y} \frac{d^2 Y}{d y^2} \right)$$

Let:

$$\frac{1}{T} \frac{d^2 T}{d t^2} = -\lambda \text{(temporal part)}$$

$$\frac{1}{X} \frac{d^2 X}{d x^2} + \frac{1}{Y} \frac{d^2 Y}{d y^2} = -\frac{\lambda}{c^2} \text{(spatial part)}$$

Split again:

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\mu,\frac{1}{Y}\frac{d^2Y}{dy^2}=-\left(\frac{\lambda}{c^2}-\mu\right)$$

Solving Each ODE

1. $X''+\mu X=0$, $X(0)=X(a)=0 \Rightarrow X_n(x)=\sin\left(\frac{n\pi x}{a}\right),\mu_n=\left(\frac{n\pi}{a}\right)^2$
2. $Y''+\nu Y=0$, $Y(0)=Y(b)=0 \Rightarrow Y_m(y)=\sin\left(\frac{m\pi y}{b}\right),\nu_m=\left(\frac{m\pi}{b}\right)^2$
3. $T''+\lambda T=0$, where $\lambda=c^2(\mu_n+\nu_m) \Rightarrow T_{nm}(t)=A_{nm}\cos(\omega_{nm}t)+B_{nm}\sin(\omega_{nm}t)$, with $\omega_{nm}=\sqrt{\lambda}=c\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}$

19.6 General Solution

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{nm}\cos(\omega_{nm}t)+B_{nm}\sin(\omega_{nm}t)\right]\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

The coefficients A_{nm}, B_{nm} are determined using initial conditions via **double Fourier sine series**.

19.7 Normal Modes and Natural Frequencies

Each pair (n,m) corresponds to a **normal mode** with associated **natural frequency**:

$$\omega_{nm}=c\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}$$

The **fundamental mode** occurs at $n=1,m=1$, with lowest frequency.

19.8 Examples of Membrane Vibration

Example 1: Square Membrane

Let $a=b$. The natural frequencies simplify to:

$$\omega_{nm} = \frac{c\pi}{a}\sqrt{n^2+m^2}$$

Modes such as (1,1), (2,1), (1,2), etc., show **symmetric and asymmetric patterns** of vibration.

Example 2: Initial Displacement Only

Suppose $u(x,y,0)=f(x,y)$, $\frac{\partial u}{\partial t}(x,y,0)=0$. Then all $B_{nm}=0$, and:

$$A_{nm} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)dx\,dy$$

19.9 Applications in Civil Engineering

- **Structural dynamics** of slabs and floor systems.
- **Seismic analysis** of structures (vibration modeling).
- **Membrane structures** like tensile roofs.
- **Sound propagation and acoustic insulation** in building design.
- **Vibration isolation** in bridges and tall buildings.

19.10 Numerical Methods for the 2D Wave Equation

While analytical solutions using separation of variables are useful for simple geometries and boundary conditions, real-world structures often demand **numerical solutions** due to complexity. Common numerical techniques include:

19.10.1 Finite Difference Method (FDM)

Discretize the domain into a grid:

Let:

- $u_{i,j}^n$ be the approximation of $u(x_i, y_j, t_n)$,
- Δx, Δy: spatial step sizes,
- Δt: time step.

The **explicit finite difference scheme** for the 2D wave equation is:

$$u_{i,j}^{n+1} = 2u_{i,j}^n - u_{i,j}^{n-1} + \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n\right) + \left(\frac{c\Delta t}{\Delta y}\right)^2 \left(u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n\right)$$

Stability Criterion (CFL condition):

$$\left(\frac{c\,\Delta t}{\Delta x}\right)^2+\left(\frac{c\,\Delta t}{\Delta y}\right)^2\leq 1$$

This ensures that the solution remains stable over time.

19.10.2 Finite Element Method (FEM)

FEM is more powerful for irregular domains. The basic idea involves:

- Dividing the domain into **elements (triangles/quads)**.
- Approximating the solution using **basis functions** over each element.
- Applying **Galerkin's method** to convert the PDE into a system of algebraic equations.

In civil engineering software (e.g., ANSYS, SAP2000), FEM is widely used to model membrane behavior under loads.

19.11 Effects of Damping

In practice, all materials exhibit **damping**, i.e., gradual loss of vibrational energy. The 2D wave equation is modified as:

$$\frac{\partial^2 u}{\partial t^2}+2\beta\frac{\partial u}{\partial t}=c^2\nabla^2 u$$

where β is the **damping coefficient**.

Solutions in this case decay over time:

$$u(x,y,t)=e^{-\beta t}\cdot\text{(oscillating part)}$$

Engineering Relevance:

- Damping reduces **resonance risk**.
- Important in **earthquake-resistant design**.
- Applied in **vibration isolators** and **soundproof membranes**.

19.12 Circular Membrane Model (Polar Coordinates)

For membranes like **circular drums**, we use polar coordinates (r,θ). The 2D wave equation becomes:

$$\frac{\partial^2 u}{\partial t^2}=c^2\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right]$$

Assume axisymmetric vibration: $u=u(r,t)$

$$\Rightarrow\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)$$

This leads to **Bessel's equation** in space:

$$r^2\frac{d^2 R}{dr^2}+r\frac{dR}{dr}+(\lambda r^2)R=0$$

The solution involves **Bessel functions**:

$$R(r)=J_0\left(\frac{\alpha_n r}{a}\right)$$

Where α_n are zeros of J_0. This gives **natural frequencies** for circular membranes.

19.13 Experimental Visualization and Validation

In real-world civil labs, **Chladni plate experiments** help visualize mode shapes:

- A thin plate or membrane is sprinkled with sand and vibrated using a speaker.
- Sand accumulates along **nodal lines**, forming intricate patterns.

These patterns validate theoretical mode shapes of $u(x,y,t)$.

Engineers also use:

- **Laser Doppler Vibrometers** to measure vibrations.
- **Accelerometers** and **strain gauges** for dynamic testing.

19.14 Software Tools for Membrane Simulation

Several software platforms support 2D wave simulations:

Tool	Application
MATLAB	PDE Toolbox, visualization of $u(x, y, t)$
COMSOL Multiphysics	Finite element modeling of wave propagation
ANSYS	Structural vibration and membrane dynamics
ABAQUS	Modal and harmonic response analysis
Python	NumPy/SciPy + Matplotlib for custom PDE solvers

19.15 Real-World Applications in Civil Engineering

Application	Description
Tensile Structures	Design of lightweight tensile roofs (stadiums, airports)
Seismic Engineering	Vibration response of floors, foundations, bridges
Acoustic Engineering	Design of membrane-based sound absorbers
Smart Structures	Embedded sensors in membranes to detect vibrations
Biomedical Structures	Modeling artificial membranes in implants