Chapter 1: Linear Differential Equations

1.1 Introduction

In the field of Civil Engineering, mathematical models are crucial to analyze structures, predict physical behaviors, and solve engineering problems. One of the most powerful tools in this modeling is **Differential Equations**, particularly **Linear Differential Equations**, which describe how a particular quantity changes over time or space. From analyzing fluid flow in pipelines, heat conduction in concrete, to deflection in beams and vibrations in structures, these equations form the foundation of mathematical simulation in engineering.

This chapter introduces **first-order** and **second-order linear differential equations**, their classifications, standard methods of solution, and their application in engineering problems.

1.2 Basic Concepts and Terminology

1.2.1 Differential Equation

A **differential equation** is an equation that involves an unknown function and its derivatives.

1.2.2 Order and Degree

- **Order**: The highest derivative present in the equation.
- **Degree**: The power of the highest derivative (provided the equation is polynomial in derivatives).

1.2.3 Linear Differential Equation

A differential equation is said to be **linear** if the dependent variable and its derivatives appear to the first power and are not multiplied together.

For example:

- **First-order linear DE**:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

- **Second-order linear DE**:

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

1.3 First-Order Linear Differential Equations

1.3.1 General Form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

1.3.2 Solution Method: Integrating Factor (IF)

The standard method to solve is:

1. Multiply both sides by an integrating factor:

$$\mu(x) = e^{\int P(x)\,dx}$$

2. This transforms the equation into:

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$$

3. Integrate both sides:

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,dx + C\right]$$

1.3.3 Example

Solve:

$$\frac{dy}{dx} + 2y = e^{-x}$$

Solution:

- $P(x) = 2$, so $\mu(x) = e^{\int 2dx} = e^{2x}$
- Multiply both sides:

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{x} \Rightarrow \frac{d}{dx}(e^{2x}y) = e^{x}$$

- Integrate:

$$e^{2x}y = \int e^{x}\,dx = e^{x} + C \Rightarrow y = e^{-x} + Ce^{-2x}$$

1.4 Second-Order Linear Differential Equations

1.4.1 General Form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

This can be **homogeneous** (when $R(x) = 0$) or **non-homogeneous**.

1.5 Homogeneous Equations with Constant Coefficients

1.5.1 General Form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

1.5.2 Auxiliary Equation (AE)

$$am^2 + bm + c = 0$$

Solve for roots m_1, m_2 and classify:

Case I: Real and distinct roots m_1, m_2

$$y = C_1e^{m_1x} + C_2e^{m_2x}$$

Case II: Real and equal roots m

$$y = (C_1 + C_2x)e^{mx}$$

Case III: Complex roots $m = \alpha \pm i\beta$

$$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$$

1.5.3 Example

Solve:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Solution:

- AE: $m^2 - 5m + 6 = 0 \Rightarrow m = 2, 3$
- General solution:

$$y = C_1e^{2x} + C_2e^{3x}$$

1.6 Non-Homogeneous Linear Equations

1.6.1 General Form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = R(x)$$

1.6.2 Complete Solution

$$y = y_c + y_p$$

- y_c: Complementary function (solution of homogeneous equation)
- y_p: Particular solution

1.7 Methods of Finding Particular Solution

1.7.1 Method of Undetermined Coefficients

Use when $R(x)$ is polynomial, exponential, or sinusoidal.

- Assume a suitable form for y_p
- Substitute into the equation to find constants.

1.7.2 Method of Variation of Parameters

Use when $R(x)$ is not suitable for undetermined coefficients.

If the homogeneous solution is:

$$y_c = C_1 y_1(x) + C_2 y_2(x)$$

Then particular solution is:

$$y_p = u_1(x) y_1(x) + u_2(x) y_2(x)$$

Where u_1, u_2 are found by solving:

$$\begin{cases} u_{1'} y_1 + u_{2'} y_2 = 0 \\ u_{1'} y_{1'} + u_{2'} y_{2'} = R(x) \end{cases}$$

1.8 Fourier Transform (Overview in Engineering Context)

In later chapters, you will learn about Fourier transforms used to solve differential equations with boundary conditions—useful in heat conduction, signal processing, and structural analysis. For now, understand that these tools help convert differential equations into algebraic equations, simplifying their solution.

1.9 Applications in Civil Engineering

Area	Application
Structural Engineering	Beam deflection, load distribution
Fluid Mechanics	Flow in open channels and pipes
Geotechnical Engineering	Soil consolidation
Transportation Engineering	Vehicle motion models
Environmental Engineering	Contaminant transport, decay processes

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1.10 Summary

- Linear differential equations are classified by order and linearity.
- First-order linear DEs are solved using the **integrating factor**.
- Second-order linear DEs are solved by:
  - Finding the **complementary function** (from AE)
  - Adding a **particular solution**
- Two main methods: **Undetermined Coefficients** and **Variation of Parameters**
- Widely applicable in civil engineering problems like deflection, vibration, heat, and fluid dynamics.
