Solid Mechanics
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Lecture - 12
Longitudinal and Shear Strains

Hello everyone! Welcome to Lecture 12! In the previous lecture, we had begun with the formulation for longitudinal strain. In this lecture, we will finish that and then discuss about shear strain.

1 Longitudinal Strain (contd.) (start time: 00:19)

In the last lecture, we had derived the following expression for stretch (λ):

$$\lambda^2(\underline{X}, \underline{n}) = \underline{n} \cdot \left[\underline{\underline{I}} + \nabla \underline{u} + \nabla \underline{u}^T + \nabla \underline{u}^T \nabla \underline{u}\right] \underline{n} + O(||\Delta \underline{X}||). \tag{1}$$

We finally need to reduce the length of the line element in the reference configuration to zero in order to obtain the local value of stretch, i.e., we have to take the limit of $||\Delta \underline{X}|| \to 0$. The $O(||\Delta \underline{X}||)$ then vanishes. We finally have

$$\lambda^2(\underline{X}, \underline{n}) = \underline{n} \cdot \left[\underline{\underline{I}} + \nabla \underline{u} + \nabla \underline{u}^T + \nabla \underline{u}^T \nabla \underline{u}\right] \underline{n} \tag{2}$$

We can simplify this even further. In this course, we will only be working with displacements such that their gradients are very small, i.e.,

$$\left|\frac{\partial u_i}{\partial x_j}\right| \ll 1 \tag{3}$$

The term $\nabla \underline{u}^T \nabla \underline{u}$ in (2), when written in matrix form, has two matrices multiplied and each of them contains derivatives of u. So, the product matrix will have components which are quadratic combinations of displacement gradients. As the displacement gradients themselves are very small, their quadratic combinations will be even smaller. Hence, the term $\nabla \underline{u}^T \nabla \underline{u}$ becomes insignificant compared to $\nabla \underline{u}$ and can be neglected. One may think that as $\nabla \underline{u}$ itself is much smaller than 1, it should also be neglected. But, in the definition for strain, it will turn out to be the most significant term (see equation (6) below). The idea is that we keep the most significant term while neglecting the other less significant terms. So finally, we have the following expression:

$$\begin{aligned}\lambda^2(\underline{X}, \underline{n}) &\approx \underline{n} \cdot \left[\underline{\underline{I}} + \nabla \underline{u} + \nabla \underline{u}^T\right] \underline{n} \\ &= \underline{n} \cdot \left[\underline{n} + \left(\nabla \underline{u} + \nabla \underline{u}^T\right) \underline{n}\right] \\ &= 1 + \left[\left(\nabla \underline{u} + \nabla \underline{u}^T\right) \underline{n}\right] \cdot \underline{n}\end{aligned} \tag{4}$$

Having obtained longitudinal stretch, the longitudinal strain then becomes

$$\begin{aligned}\epsilon(\underline{X},\underline{n}) &= \lambda - 1 \\ &= \sqrt{1+\left[\left(\underline{\nabla}\,\underline{u}+\underline{\nabla}\,\underline{u}^T\right)\underline{n}\right]\cdot\underline{n}} - 1\end{aligned} \tag{5}$$

We can now use binomial expansion to simplify the square root term which yields the following simpler tensor formula for longitudinal strain at a point in a prescribed direction

$$\epsilon(\underline{X},\underline{n}) \approx 1+\frac{1}{2}\left[\left(\underline{\nabla}\,\underline{u}+\underline{\nabla}\,\underline{u}^T\right)\underline{n}\right]\cdot\underline{n} - 1 = \frac{1}{2}\left[\left(\underline{\nabla}\,\underline{u}+\underline{\nabla}\,\underline{u}^T\right)\underline{n}\right]\cdot\underline{n} \tag{6}$$

Here again, we have dropped quadratic and higher order terms from binomial expansion for the same reason as earlier.

1.1 Longitudinal strains along coordinate axes (start time: 10:17)

If the line element direction ($\underline{n}$) is taken to be $\underline{e}_1$, the matrix form of (6) in ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$) coordinate system will be

$$\begin{aligned}\epsilon(\underline{X},\underline{e}_1) &= \frac{1}{2}\left[\left(\begin{bmatrix}\frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3}\\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3}\\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3}\end{bmatrix}+\begin{bmatrix}\frac{\partial u_1}{\partial X_1} & \frac{\partial u_2}{\partial X_1} & \frac{\partial u_3}{\partial X_1}\\ \frac{\partial u_1}{\partial X_2} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_3}{\partial X_2}\\ \frac{\partial u_1}{\partial X_3} & \frac{\partial u_2}{\partial X_3} & \frac{\partial u_3}{\partial X_3}\end{bmatrix}\right)\begin{bmatrix}1\\0\\0\end{bmatrix}\right]\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} \\ &= \frac{\partial u_1}{\partial X_1}.\end{aligned} \tag{7}$$

Similar analysis can be done for $\underline{n} = \underline{e}_2, \underline{e}_3$ leading to

$$\epsilon(\underline{X},\underline{e}_1) = \frac{\partial u_1}{\partial X_1}, \quad \epsilon(\underline{X},\underline{e}_2) = \frac{\partial u_2}{\partial X_2}, \quad \epsilon(\underline{X},\underline{e}_3) = \frac{\partial u_3}{\partial X_3} \tag{8}$$

We can compare the strains obtained above with what we have seen in our schools. For example, consider the straight bar that we had seen in the last lecture (see Figure 1). The origin of the coordinate system is at the clamped end and the bar length is along x-axis. If we say the bar gets uniformly stretched by λ or the bar gets axially strained by ϵ, the displacement of the bar will be given by

$$u = (\lambda - 1)x = \epsilon x. \tag{9}$$

This is because displacement will be zero at the clamped end and then increases linearly along the length of the bar. When we calculate $\frac{\partial u}{\partial x}$ from the above equation, we get

$$\frac{\partial u}{\partial x} = \lambda - 1 = \epsilon. \tag{10}$$

So, we can see that our formula for strain $\frac{\partial u}{\partial x}$ yields correct strain value. This completes our discussion for longitudinal strain in a general three-dimensional body.

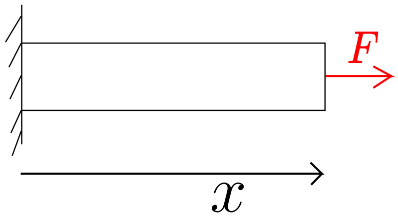

Figure 1: A straight bar with a force applied along the length of the bar.

2 Shear strain (start time: 15:41)

We have another type of strain which measures the change in angle between two line elements which are perpendicular in the reference configuration (also see Figure 2). Such a measure of strain is called shear strain. After the body gets deformed, the angle between the line elements changes and need not be 90° anymore. If the angle between them after deformation is denoted by β, the change in angle (denoted by α) will be

$$\alpha = 90^\circ - \beta \tag{11}$$

which is the shear strain at this point in the body. It generates distortion in a body leading to change in its shape whereas longitudinal strain changes the size of the body. For example, a rectangular body can become a parallelogram (keeping its area unchanged) due to shear strain as the angle between its edges changes.

2.1 Formulation (start time: 19:09)

Our goal is to find a mathematical expression for shear strain just as we found one for longitudinal strain. Consider a body before and after deformation as shown in Figure 2. At the point of interest $\underline{X}$ in the reference configuration, we identify two perpendicular line elements $\Delta\underline{X}$ and $\Delta\underline{Y}$. After deformation of the body, $\Delta\underline{X}$ becomes $\Delta\underline{x}$ and $\Delta\underline{Y}$ becomes $\Delta\underline{y}$. Let the unit vectors along line elements $\Delta\underline{X}$ and $\Delta\underline{Y}$ be denoted by $\underline{n}$ and $\underline{m}$ respectively, i.e.,

$$\underline{n} = \frac{\Delta\underline{X}}{||\Delta\underline{X}||}, \quad \underline{m} = \frac{\Delta\underline{Y}}{||\Delta\underline{Y}||} \tag{12}$$

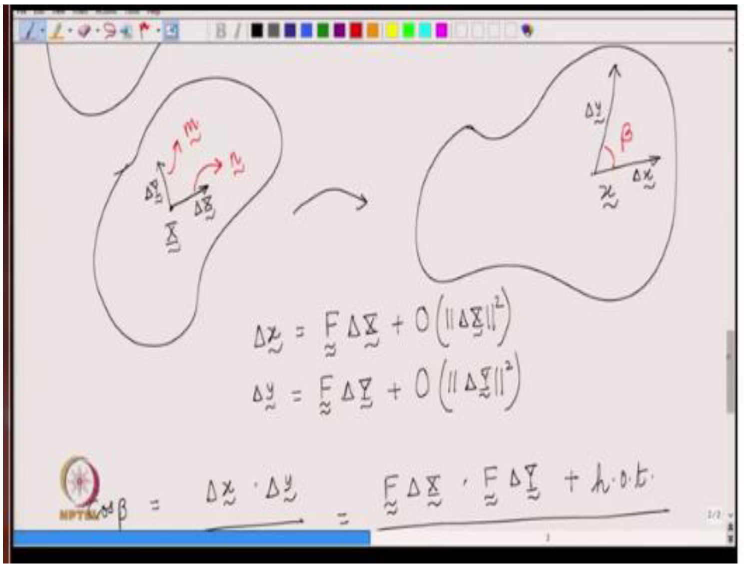

Figure 2: Two initially perpendicular line elements at a point $\underline{X}$ in the body are shown on the left. The right figure shows the deformed body with the angle between the line elements changed.

In general, the two line elements undergo stretching and the angle between them also changes. We have already derived the expression for deformed line element in terms of the deformation gradient te nsor ($\underline{\underline{F}}$) according to which

$$\begin{aligned} \Delta \underline{x} &= \underline{\underline{F}} \Delta \underline{X} + O(||\Delta \underline{X}||^2) \\ \Delta \underline{y} &= \underline{\underline{F}} \Delta \underline{Y} + O(||\Delta \underline{Y}||^2) \end{aligned} \tag{13}$$

Similar to the case of longitudinal strain, we keep the higher order terms because we don't know if they are significant or not. The angle between these two line elements in deformed configuration (denoted by β) will be given by

$$cos\beta = \frac{\Delta \underline{x} \cdot \Delta \underline{y}}{||\Delta \underline{x}|| \, ||\Delta \underline{y}||} \tag{14}$$

Let us substitute equation (13) in the numerator above and express the terms in denominator using stretch, i.e.,

$$\begin{aligned} cos\beta &= \frac{\underline{\underline{F}} \Delta \underline{X} \cdot \underline{\underline{F}} \Delta \underline{Y} + O(||\Delta \underline{X}|| \; ||\Delta \underline{Y}||^2) + O(||\Delta \underline{X}||^2 ||\Delta \underline{X}||)}{\lambda(\underline{X}, \underline{n}) ||\Delta \underline{X}|| \quad \lambda(\underline{X}, \underline{m}) ||\Delta \underline{Y}||} \\ &= \frac{\underline{\underline{F}} \dfrac{\Delta \underline{X}}{||\Delta \underline{X}||} \cdot \underline{\underline{F}} \dfrac{\Delta \underline{Y}}{||\Delta \underline{X}||}}{(1 + \epsilon(\underline{X}, \underline{n}))\,(1 + \epsilon(\underline{X}, \underline{m}))} + O(||\Delta \underline{X}||) + O(||\Delta \underline{Y}||) \\ &= \frac{\underline{\underline{F}}\, \underline{n} \cdot \underline{\underline{F}}\, \underline{m}}{(1 + \epsilon(\underline{X}, \underline{n}))\,(1 + \epsilon(\underline{X}, \underline{m}))} + h.o.t. \quad \text{(using (12))} \end{aligned} \tag{15}$$

Here *h.o.t.* denotes all the higher order terms which would vanish in the limit of length of both the line elements going to zero. As $\epsilon(\underline{X},\underline{n})$ and $\epsilon(\underline{X},\underline{m})$ are along directions $\underline{n}$ and $\underline{m}$ respectively, we denote them as ϵ_{nn} and ϵ_{mm}, i.e.,

$$cos\beta \quad = \quad \frac{\underline{\underline{F}}\,\underline{n}\cdot\underline{\underline{F}}\,\underline{m}}{(1+\epsilon_{nn})\,(1+\epsilon_{mm})} = \underline{\underline{F}}\,\underline{n}\cdot\underline{\underline{F}}\,\underline{m}\,(1+\epsilon_{nn})^{-1}(1+\epsilon_{mm})^{-1} \tag{16}$$

We can now apply binomial expansion formula because we know that longitudinal strains are very small compared to 1 since they are of the order of the displacement gradients, i.e.,

$$cos\beta \approx (\underline{\underline{F}}\,\underline{n}\cdot\underline{\underline{F}}\,\underline{m})\,(1-\epsilon_{nn})(1-\epsilon_{mm}) + h.o.t. \tag{17}$$

Upon further expressing the deformation gradient tensor in terms of displacement gradient, we have

$$cos\beta = (\underline{\underline{I}}+\nabla\,\underline{u})\underline{n}\cdot(\underline{\underline{I}}+\nabla\,\underline{u})\underline{m}\,(1-\epsilon_{nn})(1-\epsilon_{mm}) \tag{18}$$

We can now use the following identity (proved in previous lecture):

$$\underline{\underline{A}}\,\underline{a}\cdot\underline{b} = \underline{a}\cdot\underline{\underline{A}}^T\underline{b} \tag{19}$$

to bring the $(\underline{\underline{I}} + \nabla\,\underline{u})$ term to the other side of the dot product, i.e.,

$$\begin{aligned} cos\beta \quad &= \quad \underline{n}\cdot\left[(\underline{\underline{I}}+\nabla\,\underline{u})^T(\underline{\underline{I}}+\nabla\,\underline{u})\,\underline{m}\right](1-\epsilon_{mm})(1-\epsilon_{nn}) \\ &= \quad \underline{n}\cdot\left[\underline{\underline{I}}+\nabla\,\underline{u}+\nabla\,\underline{u}^T+\nabla\,\underline{u}^T\nabla\,\underline{u}\right]\underline{m}\,(1-\epsilon_{mm})(1-\epsilon_{nn}) \end{aligned} \tag{20}$$

We can again drop $\nabla\,\underline{u}^T\nabla\,\underline{u}$ term as we had done in the longitudinal strain formulation leading to

$$\begin{aligned} cos\beta \quad &= \quad \underline{n}\cdot\left[\underline{m}+(\nabla\,\underline{u}+\nabla\,\underline{u}^T)\underline{m}\right](1-\epsilon_{mm})(1-\epsilon_{nn}) \\ &= \quad \left(\underline{n}\cdot\underline{m}+\left[(\nabla\,\underline{u}+\nabla\,\underline{u}^T)\underline{m}\right]\cdot\underline{n}\right)(1-\epsilon_{mm})(1-\epsilon_{nn}) \end{aligned} \tag{21}$$

As $\underline{m}$ and $\underline{n}$ are perpendicular, their dot product will be zero. Also, as longitudinal strains are much less than 1, the terms $(1 - \epsilon_{mm})$ and $(1 - \epsilon_{nn})$ can be approximated to be 1. Thus, we finally have

$$cos\beta = \left[(\nabla\,\underline{u}+\nabla\,\underline{u}^T)\underline{m}\right]\cdot\underline{n} + h.o.t. \tag{22}$$

Using basic trigonometry and equation (11), we can then write

$$cos\beta = sin(90^\circ-\beta) = sin(\alpha) = (\nabla\,\underline{u}+\nabla\,\underline{u}^T)\underline{m}\cdot\underline{n} \tag{23}$$

As the gradients of the displacement are very small, the RHS of the above equation is very small (usually of the order of 10^{-2}). Therefore $sin(\alpha)$ also becomes very small and can be replaced with just α, i.e.,

$$\boxed{\alpha = (\nabla \underline{u} + \nabla \underline{u}^T)\underline{m} \cdot \underline{n}} \tag{24}$$

We can notice that the above formula for shear strain depends on the directions of the two line elements. Thus, at a point in the body, we will have different value of shear strain for different pairs of perpendicular line elements.

3 Significance of $[\frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)]$ (start time: 33:06)

Let us write the matrix form of $\frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)$ in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system, i.e.,

$$\left[\tfrac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2} + \dfrac{\partial u_2}{\partial X_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3} + \dfrac{\partial u_3}{\partial X_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2} + \dfrac{\partial u_2}{\partial X_1}\right) & \dfrac{\partial u_2}{\partial X_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3} + \dfrac{\partial u_3}{\partial X_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3} + \dfrac{\partial u_3}{\partial X_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3} + \dfrac{\partial u_3}{\partial X_2}\right) & \dfrac{\partial u_3}{\partial X_3} \end{bmatrix} \tag{25}$$

Being a symmetric tensor, its matrix form also turns out to be symmetric. We can see upon comparing from equation (8) that the diagonal elements of the above matrix give us longitudinal strains along $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$ directions. To understand the significance of the off-diagonal elements, let us choose the two line element directions $\underline{n}$ and $\underline{m}$ in equation (24) to be $\underline{e}_1$ and $\underline{e}_2$ respectively. We denote the shear strain for this case as α_{12} because of the directions chosen which equals

$$\begin{aligned} \alpha_{12} &= (\nabla \underline{u} + \nabla \underline{u}^T)\underline{e}_1 \cdot \underline{e}_2 \\ &= 2\Big(\frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)\Big)\underline{e}_1 \cdot \underline{e}_2 \end{aligned} \tag{26}$$

When worked out using the matrix form, it turns out to be twice of the off-diagonal term in second row and first column of (25), i.e.,

$$\alpha_{12} = \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \tag{27}$$

We can thus conclude that in matrix (25), the off-diagonal elements represent half the shear strains.

3.1 Geometric interpretation of shear strain formula

Let us visualize the above expression for shear strain. Consider the body shown in Figure 3 and try to obtain shear strain at point $\underline{X}$. We choose the line elements along $\underline{e}_1$ and $\underline{e}_2$ in the reference configuration having their lengths ΔX_1 and ΔX_2 respectively. After deformation, the point $\underline{X}$ as well as the line vectors will shift to new positions as shown in the figure. We have also drawn horizontal and vertical axes (shown by dotted lines) at the deformed position $\underline{x}$. These lines will help us to evaluate the angle change. The total change in angle of the two line elements will be the sum of the angles shown in blue and green. We call the angle shown in blue as α_b and the angle shown in green as α_g. Thus

$$\alpha = \alpha_b + \alpha_g \tag{28}$$

Let's find α_b first. Consider the right-angled triangle containing angle α_b. The red distance there is the difference in y-displacement of the tip and vertex of the initially horizontal line element.

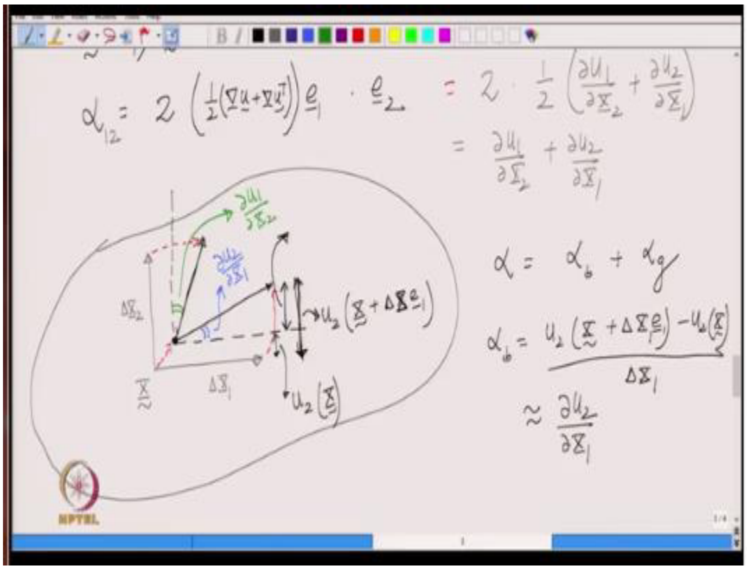

Figure 3: An arbitrary body with angles between the reference line elements and deformed line elements shown.

The length of the base of this right angled triangle will be approximately ΔX_1 since the longitudinal strain is much smaller than 1. Thus, the angle α_b will be given by

$$\alpha_b = tan^{-1}\left(\frac{u_2(\underline{X} + \Delta X_1\ \underline{e}_1) - u_2(\underline{X})}{\Delta X_1}\right) \approx \frac{u_2(\underline{X} + \Delta X_1\ \underline{e}_1) - u_2(\underline{X})}{\Delta X_1} \tag{29}$$

Finally, to find this angle at point $\underline{X}$ itself, ΔX_1 should tend to zero, i.e.,

$$\alpha_b = \lim_{\Delta X_1 \to 0} \frac{u_2(\underline{X} + \Delta X_1 \ \underline{e}_1) - u_2(\underline{X})}{\Delta X_1} = \frac{\partial u_2}{\partial X_1} \tag{30}$$

By similar analysis, we can find α_g as

$$\alpha_g = \lim_{\Delta X_2 \to 0} \frac{u_1(\underline{X} + \Delta X_2 \ \underline{e}_2) - u_1(\underline{X})}{\Delta X_2} = \frac{\partial u_1}{\partial X_2} \tag{31}$$

Using equation (28), we then get

$$\alpha = \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_2} \tag{32}$$

Thus we have realized the physical significance of this expression for shear strain. It is the sum of the change in angles of the two line elements from their initial orientation.