Discrete Mathematics
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Module No # 07
Lecture No # 31
Tutorial 5

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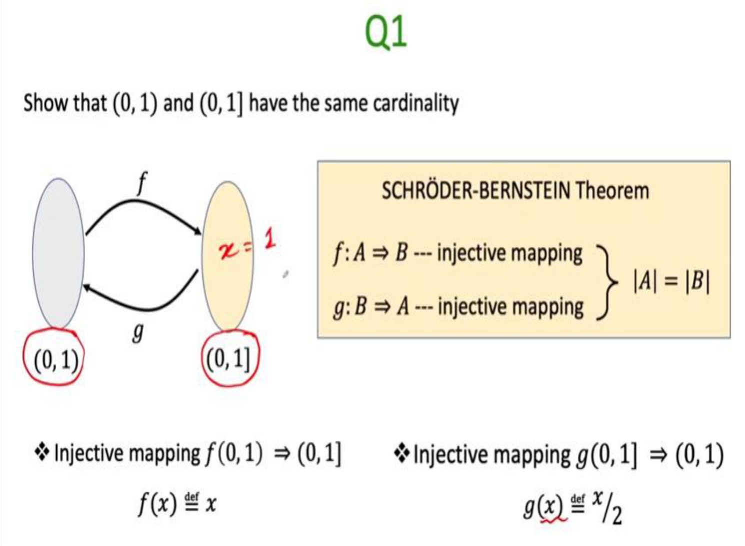

Hello everyone welcome to tutorial number 5 so let us start with question number 1 we have to show in question number 1 that these two sets have the same cardinality. So the first set here is the set of all real numbers between (0,1) but excluding 0 as well as 1. Whereas the second collection here it is also the set of all real numbers between (0, 1] but 1 is inclusive that means 1 is allowed. That is why the square bracket here and 0 is not allowed.

So that is the interpretation of these two sets we have to prove that these two sets have the same cardinality. How we can do that so recall the Schroder-Bernstein theorem which says that, if you want to prove that two sets have same cardinality show injective mappings from the first set to the second and from the second set to the first. So we are going to do the same thing here so here are our two sets.

Let us consider the injective mapping f which is the identity mapping so clearly this mapping is an injective mapping from this set to this set. Because you take any two different real numbers x and y the corresponding image will be x and y and they will be different and they will be in the range which is allowed as per function f. So your domain is not allowed to have the numbers 0 and 1 and so is the images.

Now if I want to take the injective mapping in the reverse direction then consider the injective mapping g defined to be x / 2 that means. If you want to find out the value of g(x) the output is (x / 2). So that means if your x here which we are considering is different from 1 then clearly that will fall in the range (0, 1). And excluding 0 and 1 but if your x = 1 as well then the image of 1 will be 0.5 which is well within the allowed limit.

And it is again easy to verify that your mapping g is an injective mapping in the sense we have shown here two injective mappings so we can conclude that the cardinality of these two sets are the same.

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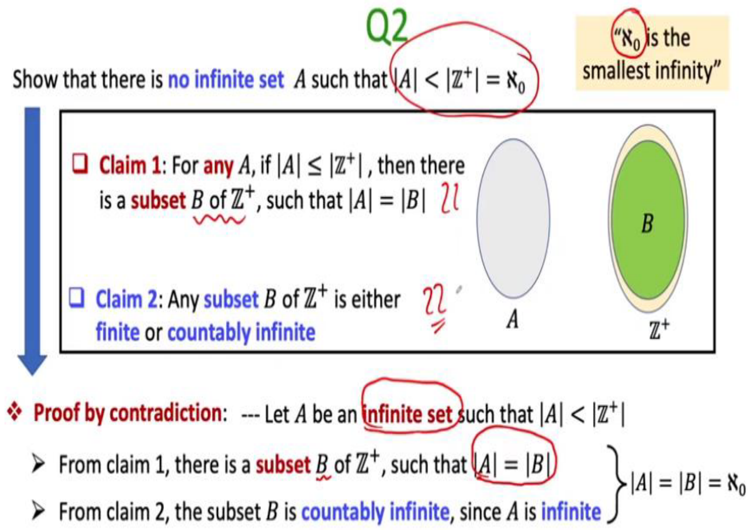

In question number 2 we have to prove that there is no infinite set A whose cardinality is strictly less than $\aleph_0$. That means its cardinality is strictly less than the cardinality of set of positive integers. So in some sense we want to prove here that is $\aleph_0$ is the smallest infinity here. So to prove this statement we will use 2 claims and we will assume for the moment that these two claims are correct and later on we will focus our attention on proving these two claims.

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The first claim is that if you have any set A whose cardinality is less than equal to the cardinality of set of positive integers then you can always find a subset of the set of positive integers which has the same cardinality as your set A. So pictorially what I am saying here is that you may have a set A whose cardinality is less than equal to the cardinality of this set Z^+. What I am saying here is that in this claim the claim says that you can always find a subset B within the set Z^+ whose size is exactly the same as the size of A.

And claim 2 is that any subset of the set of positive integers is either finite or has the same cardinality as $\aleph_0$, you cannot have any other third category of subset of the set of positive integers. So for the moment assume that these 2 claims are true let us come back and prove the statement that we are interested to prove and the proof will be by contradiction. So we want to prove that is no infinite set A satisfying this condition.

So we will assume that on contrary suppose you have an infinite set A whose cardinality is strictly less than $\aleph_0$. Now by applying claim 1 on that set A we also know that there exist some subset B of the set of positive integers whose cardinality is same as the cardinality of your A set. And now if I apply the claim 2 on that subset B I know that the subset B has to be countably infinite.

Because the subset B is a subset of the set of positive integers; and it will be either finite or countably infinite. So B is the definitely not finite because the cardinality of B is same as the cardinality of A and we are assuming here that A is an infinite set. So we are left with the second category here that means the cardinality of the set B is countably infinite. But we also know that the cardinality of B is same as cardinality of A that means the cardinality of A is also $\aleph_0$ because B is countably infinite.

So the cardinality of B is $\aleph_0$ that means the cardinality of A is also $\aleph_0$ which is a contradiction. So now the proof boils down to how exactly we prove these two claims; they are very simple.

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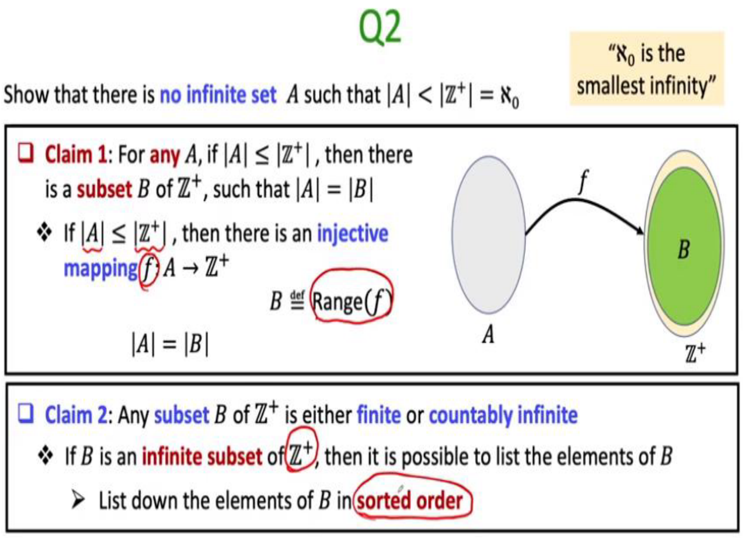

So let us first prove claim number 1; since the cardinality of A is less than equal to the cardinality of Z^+ then as per the definition of cardinalities we know that there exist an injective mapping from the set A to the set of positive integers. Then only the cardinality of A can be less than equal to the cardinality of Z^+ so based on that injective mapping f I am going to show you the existence of the required set B.

What do I do basically is, I just focus on the range of the function that means I pick up the set of images of this function f that means all the valid images as per this function f and since my function f is an injective mapping each element of A will have a unique image. So how many images I will pick? I will pick same number of the images as I have the number of elements in the domain that means the range set of f namely the set B will have the same cardinality as the cardinality of your A set.

So that is the very simple proof for claim number 1. For claim number 2 we have to prove any subset of Z^+ is either finite or countably infinite. So if we pick a finite subset of Z^+ then the proof is trivial. So let us focus on the case when we have chosen an infinite subset of Z^+. Then the idea here is that we know that Z^+ is countably infinite that means it is possible to list down all the elements of Z^+ and B is a subset of Z^+.

That means what I can do is, I can always say that it is possible to list down the elements of B why? You just list down the elements of B in sorted order, that is all, because B is the subset of

Z^+ and it is always possible to list down all the elements of Z^+. So depending upon which elements are there in B or which elements are not there in B, in the sequencing of Z^+ remove first all the elements which are not there in B. And if you arrange all those remaining elements in a sorted order then that gives you a valid listing of the elements of the set B.

And if we have the valid listing of the elements of an infinite set then that is a countably infinite set. So that shows that your subset B is countably infinite which proves your claim.

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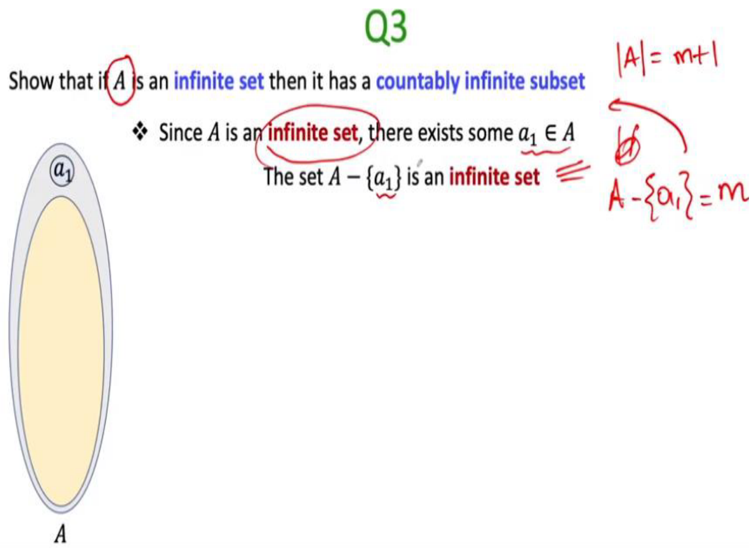

In question number 3 we are asked to show that if you are given an infinite set then it does not matter whether A is countable or uncountable,you can always find the subset of A which is countably infinite. So of course if my set A itself is countably infinite then the subset would be the set itself but in the statement I am asking you to prove this even if the set A is uncountable. So even if my set A is the set of real numbers which is an infinite set and not countable I am asking you to prove the existence of a subset of real numbers which is countable and infinite.

So we will prove this general statement how we are going to prove this? So since A is an infinite set it will have at least one element arbitrarily I pick that element I call it as a_1. Now my claim is that if I remove that element a_1 from the set A, the remaining set is still an infinite set why? Because if this remaining set $A – a_1$ if this is finite and if it is cardinality is say m then what can I say about the cardinality of A.

I can say the cardinality of A is m+1 which is a finite quantity but that goes against my assumption that A is an infinite set. So that means if I remove the elements a_1 from the set A I will still left with an infinite set.

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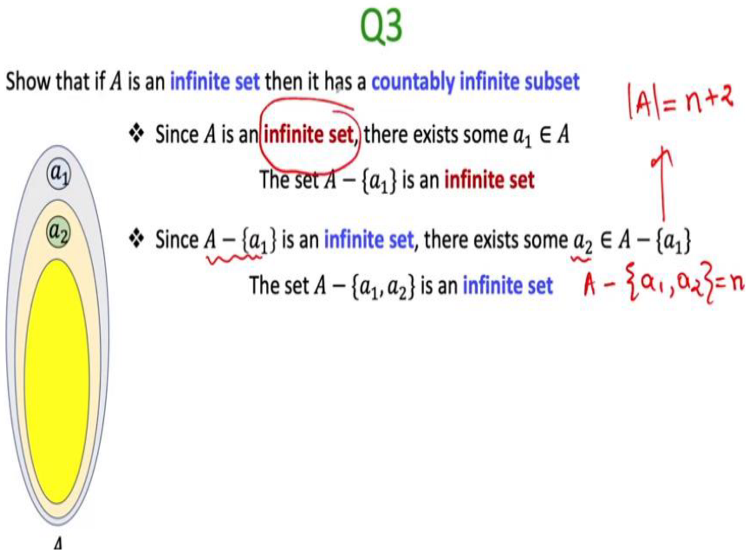

Now I again apply the same argument. I focus on the left over set namely the set which I obtained by removing element a_1. And my claim is that since it is an infinite set it will have at least 1 element a_2 I will pick it arbitrarily. And my claim is that if I now remove the element a_2 also I have already removed a_1 from A and now what I am saying is, even if I now remove a_2 the left over set will be an infinite set.

Again the proof follows using similar argument because if this left over set is not infinite suppose its cardinality is finite number say n then what I can say about the cardinality of A? I can say the cardinality of A will be n+2 which is a natural number, positive number. And which goes against the assumption that A is an infinite set.

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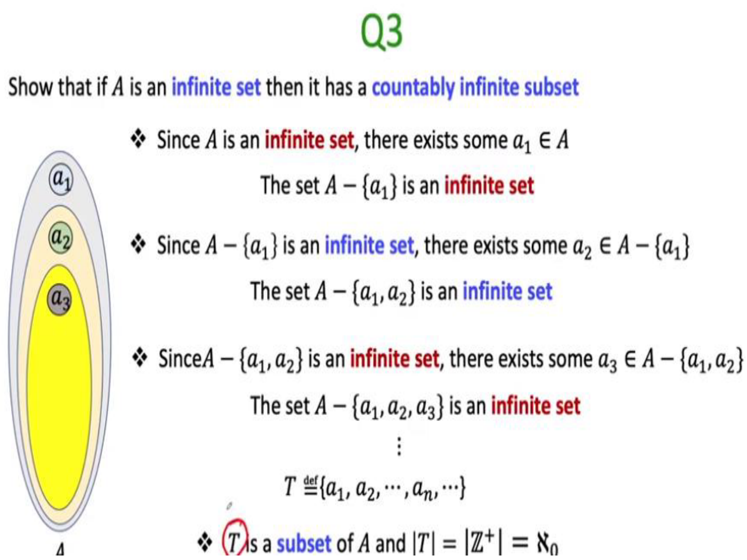

So I can keep on running this argument and what I can say now is that, in each step in each iteration the arbitrarily element which I am picking from the left over set if I list it out, if I list down all those elements, then that sequence of the elements will be or the set of those elements which I am picking in each iteration from the left over set will of course be a subset of my original set. Because in each iteration I am picking an element from the left over set and in each iteration the left over set is a subset of the original set.

And what I can say about the set of numbers which I am removing from each iteration. I can say that it is cardinality is $\aleph_0$ because I now have a valid sequencing for the elements in that subset. Namely I can arrange the elements in that set T in the order in which I have removed those elements in each iteration. So that shows that I can always find out an infinite subset from the set A whose cardinality is $\aleph_0$.

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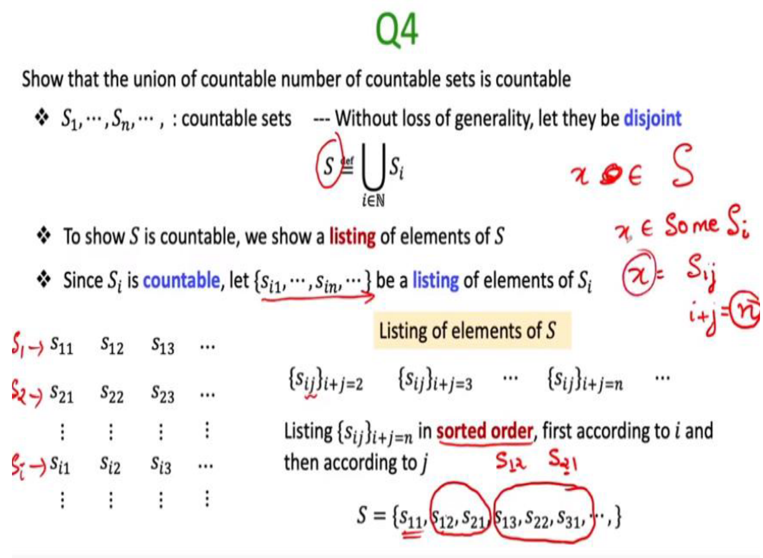

In question number 4 we want to show that if I take several countable sets and take their union namely if I perform union of countable number of countable sets then the resultant set is again countable. So imagine you are given several countable sets namely say you might be given infinite number of countable sets and without loss of generality let they be disjoint. So I call the bigger set $S \stackrel{\text{def}}{=} \bigcup_{i \in \mathbb{N}} S_i$ to be the union of all these sets S_i and I want to show that this bigger set S is also countable, if each of these individual sets were countable. So what we have to do basically is we have to show how to list down the elements of this bigger set S. So since each of the sets S_1 to S_n and S_i each of the sets S_i is countable it will have a listing of its own. So imagine that the listing of the elements of the set S_i is this. It is an infinite list and the guarantee is that each element of the set S_i will eventually occur somewhere.

So I have written down the listing of all the elements of various sets here; so this is the listing of S_1 , this is the listing of S_2 , this is the listing of S_i and so on. Now based on all these listings I have to come up with the listing of all the elements in the set S. So that I will not be missing any element and we will never get stuck infinitely while finding any element in the resultant listing. So here is the way we can list down the elements of the set S.

So if you see here within each set S_i I have used 2 indices to denote an element in the listing. First index denotes the subset number or the set number that i am focusing now. So if it is i then the first index will be i and the second index basically denotes the position of the element. Whether, it is the first element or the second element or the nth element and so on. So what I am going to do is when I want to list out the elements of the set S I will list down according to the pair of indices i, j.

And I will start by listing down all the elements of the form $S_{i,j}$ such that the summation of the 2 indices is 2; remember the minimum value of the summation of the 2 indices will be 2. Because I cannot have i, j such that i + j is 1 for that to happen at least one of the, i or j has to be 0. But I do not have any index taking the value 0 here. So the least value of the summation of the 2 indices can be 2 only.

So I will be first listing down all the elements where the summation of the indices will be 2 following by listing down all the elements whose summation of indices will be 3 and so on. Now when I am listing down all the elements whose summation of indices is n then you can have several numbers of that form whose summation of indices is n. So within that collection I will be following a sorted order, that means I will first write down according to the index i and then according to the index j.

So to demonstrate here if I focus on all indices all pair of indices i j whose summation is 2 then I have only one entry namely $S_{1\,1}$. Next I will focus on all $S_{i\,j}$'s where the summation of i and j is 3. So I have $S_{1\,2}$ and $S_{2\,1}$ and you can see that I am listing $S_{1\,2}$ first and then $S_{2\,1}$. Because i gets preference first so I will list down all the $S_{i\,j}$'s where i is equal to 1 and then followed by all $S_{i\,j}$'s where j is bigger than i.

I have 2 elements here I have $S_{1\,2}$ and $S_{2\,1}$ to list out so I will give reference to lower value of i first. So that why I have written down $S_{1\,2}$ first and then I have written down $S_{2\,1}$ then I will focus on all $S_{i\,j}$'s where the summation of i and j will be 4 and so on. So now you can see that this is a valid listing of the elements of the bigger set S you will never miss any element and you will not get stuck forever to find out an element in this listing.

Because you take any element x belonging to the bigger set S, x will belong to some S_i. You do not know which set it belongs to; that means x will be of the form $S_{i\,j}$ for some j and i + j will take some value in the set of positive integers call it n. So when you are listing down all $S_{i\,j}$'s such that the summation of i and j is n, you will be listing down the element x; that is why this is a valid listing of the elements of the set S.

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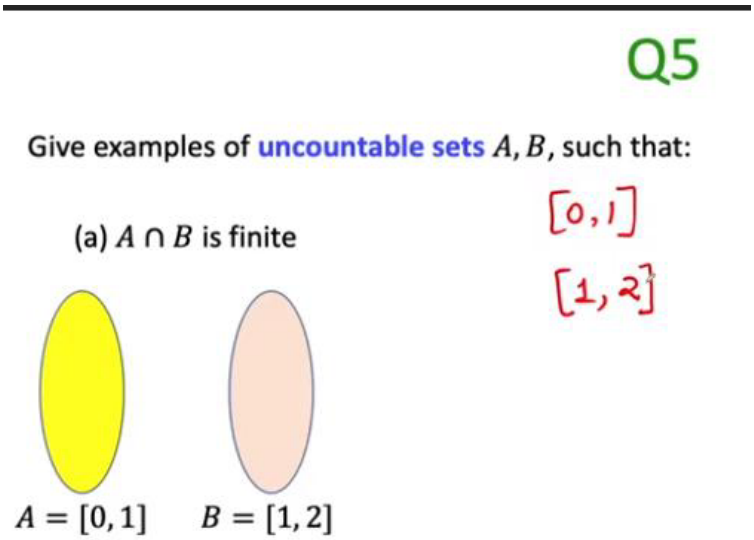

In question number 5 you are asked to give examples of uncountable sets A and B that satisfy certain conditions. So we want to find out uncountable sets such that their intersection is finite. So there can be several examples: let us take these 2 sets the set A= [0,1], the set of all the real numbers between 0 and 1 including 0 and 1. And the set B= [1,2], the set of all the real numbers between 1 and 2 including 1 and 2.

So remember we proved that a set of all real between 0 and 1, its cardinality is not $\aleph_0$. So it is an uncountable set we can follow the same argument to even prove that a set of all real numbers between 1 and 2 is also uncountable. So I am not going into the proof of that I leave it as an exercise for you.

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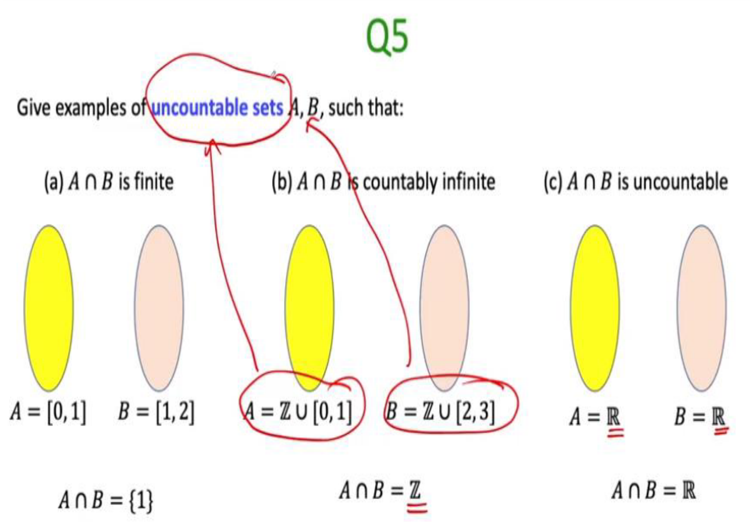

Now what can you say about intersection of this A set and B set well there is only 1 element which is there in the intersection of these 2 sets namely the number 1 and this is the singleton set and hence it is a finite set. We now want to find out examples of 2 uncountable sets such that their intersection is countably infinite. So what we can say here is, let us take the set A which is the union of the set of integers along with the set of all real numbers between 0 and 1.

And we can prove that this set is uncountable and in the same way I take the set B to be the set of all integers along with all real numbers in the range 2 and 3 including 2 and 3 also. And we can prove that both these 2 sets are uncountable. Their cardinalities are not $\aleph_0$; A is not countable; its cardinality is not $\aleph_0$ even though Z is part of that.

So Z is countable but the set of all real numbers between 0 and 1 including 0 and 1 is not countable. So that is why the overall set A becomes uncountable and similarly the overall set B also becomes uncountable. Now what can you say about their intersection? Their intersection is nothing but the set of all integers and we know that the set of integers is countable because its cardinality is $\aleph_0$.

In part C you are supposed to give uncountable sets A and B whose intersection is also uncountable and a very simple example could be take the set A and B to be an uncountable set and the same uncountable set. So if I take the set A to be the set of the real numbers and the set B also to be the set of real numbers both of them are uncountable. And clearly A ∩B will be set A itself which is the set of real numbers and which is also uncountable.

So what does this question demonstrates is that; you can have various properties among 2 uncountable sets you cannot say anything with guarantee regarding the union and intersection. Their intersection may be finite, it may be uncountable it might be countably infinite and so on.

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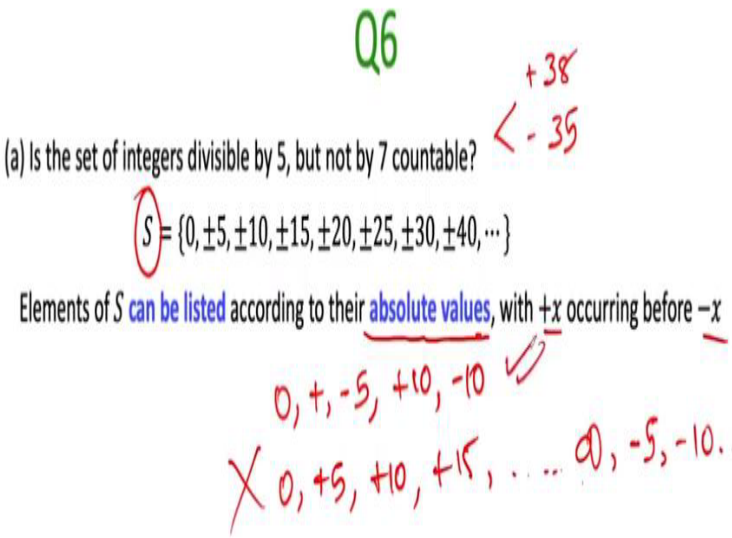

In question 6, part a, you want to find out whether the set of integers divisible by 5 but not by 7 is countable or not. So first of all what exactly is this set so let me call, denote this set by S. So the set S will have the number 0 it will have +5 – 5 it will have +10 – 10 and so on. It would not have 35 it would not have + 35 it would not have -35 because 35 is divisible by 5. But it is also divisible by 7 so are not supposed to include multiples of 7.

We are not supposed to include multiples of both 5 and 7; so now we want to show whether the set S is countable or not. So definitely S is an infinite set but the question is can we enumerate the elements of the set S. And it is easy see that we can always list down the elements of the set according to their absolute values. And if I focus on the absolute values then both +x and –x will take the same absolute value x.

So what I will do is in the listing I will write down +x and –x so basically what I am saying is that my listing here is 0 followed by +5 followed by -5 then +10 followed by -10 and so on. You cannot do the following: you cannot say that first list down all the positive things up to infinity and then followed by the minus things. This is not a valid listing because if we do this then we do not know whether we will come back ever and start listing down the negative numbers here.

Because when we start going towards the positive multiples of 5 in infinity we will never stop and there is no coming back. So that is why it is only the first listing which is a valid listing not the second one.

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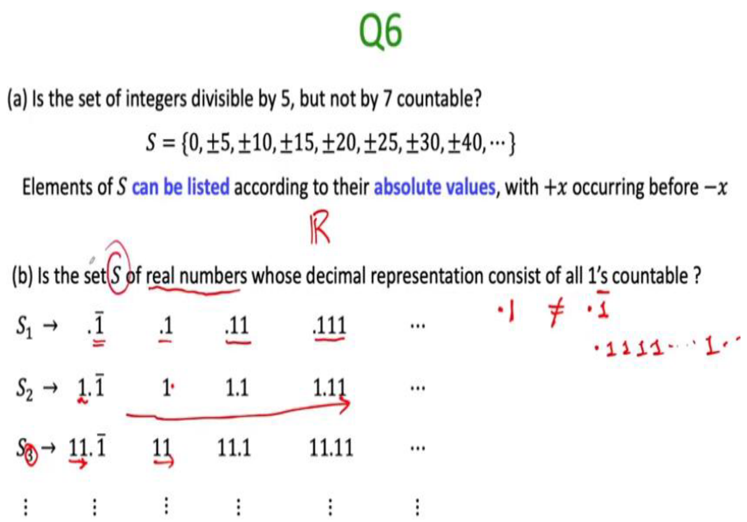

Part B we want to find out whether the set of all real numbers whose decimal representation consist of only 1s is countable or not. So remember the set of real numbers is uncountable we proved that but here we are not focusing on the set of all real numbers. We are focusing only on the real numbers whose decimal representation have only 1s and no other digit. And it turns out that this set is countable because I can view it as union of several countable sets okay.

So let S_1 denotes the real numbers consisting of only 1s where you do not have anything before the decimal point. And after the decimal point you list down all the numbers starting with real number where you have recurring 1s that means you have a series of infinite ones and occurring again and again. Followed, by the real number 0.1 followed by real number 0.11 followed by real number 0.111 and so on.

Remember 0.1 is not same as the real number where 1 is recurring. 0.1 you do not have anything after the first 1 here. But 0.1 with the recurring one denotes this real number which is clearly different from 0.1. So these are the elements in my set S_1 and I have a valid listing here. Then in S_2 what I do is, I will list down all the real number with decimal representation has only 1s and where before the decimal point I have only 1 digit.

So these are the elements in the set S_2 : 1.1, 1.11, 1.111, … In S_3 what I am going to do is I will list down all the real numbers whose decimal representation consist of only 1s , and where before the decimal point I have 2 1s, so you can imagine that when I am considering the set S_i I will be focusing on all the real numbers whose decimal representation has 1s provided there are $i - 1$ number of occurrences of 1 before the decimal point.

And after the decimal point I will give preference first to the recurring of occurrence of 1 then 0 occurrence of 1 and 1 occurrence of 1 and 2 occurrence of 1 and 3 occurrence of 1 and so on. And each of this listing is a valid listing and if I take the union of all this sets S_1, S_2, S_3 that will give me the bigger set S.

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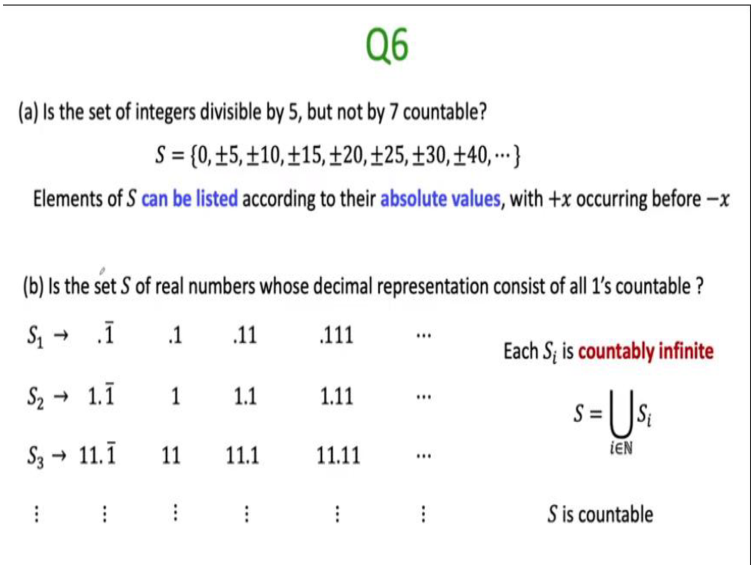

That means now I now have a valid listing here and bigger set S is the union of several countable sets and hence the bigger set S will be countably infinite. So that brings me to the end of this tutorial thank you.