Discrete Mathematics
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Module No # 06
Lecture No # 27
Countable and Uncountable Sets

Hello everyone welcome to this lecture on countable and uncountable sets.

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The plan for this lecture is as follows: in this lecture we will discuss about the cardinality of finite sets. We will discuss about the cardinality of infinite sets and we will conclude with countable and uncountable sets.

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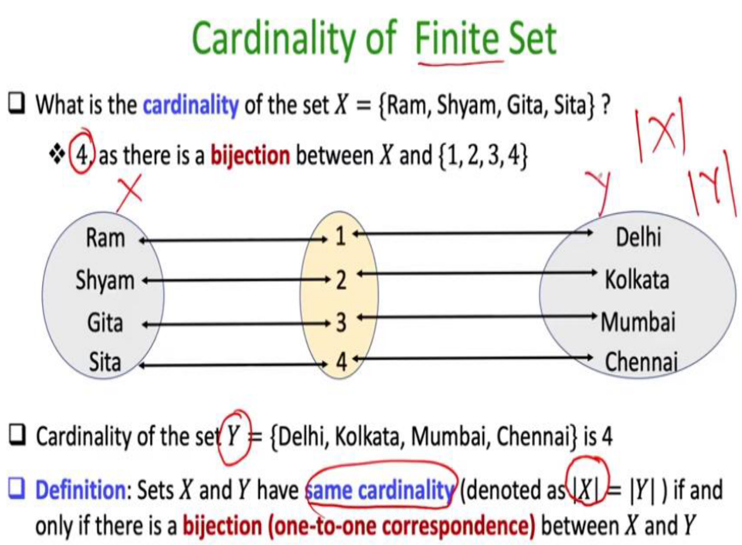

So let us begin with the cardinality of finite sets first. So if I ask you what is the cardinality of this set X with consist of the elements Ram, Sham, Gita and Sita. You will say its cardinality is 4 because it has 4 elements. Another way to put it is as follows: We can say that the cardinality of the set X is 4 because there is a bijection between the set X and the set consisting of the elements 1, 2, 3, 4. So there can be many bijections possible between the set X and the set 1, 2, 3, 4.

One of the bijection could be where the element Ram is mapped to 1, Sham is mapped to 2, Gita is mapped to 3 and Sita is mapped to 4. You can have a bijection where Ram is mapped to 2 Sham is mapped to 1 and so on that is also a possible bijection. But since there is a bijection between Ram, Sham, Gita and Sita and the set 1, 2, 3, 4 we can say that the cardinality of the set X is 4.

Because I can say that whichever element from the set X is mapped to element 1 that is the first element of set X whichever element from the set X is mapped to element 2. That is the second element of set X and so on, that is why the cardinality of the set X is 4. Now due to the same reason if I consider another set Y consisting of the elements Delhi, Kolkata, Mumbai and Chennai, its cardinality is also 4.

Because it has 4 elements or another way to put it around is there is a bijection between the set Y and the set 1, 2, 3, 4. So based on this example we can formulate the following definition we can say that: two sets X and Y have same cardinality and for that we use this notation |X|. So remember this notation denotes the cardinality of X : |X| and this notation denotes the cardinality of Y : |Y|. So if the sets X and sets Y have the same cardinality we use this notation that their cardinalities are equal. And when can we say that their cardinality are the same we can say that provided there is a bijection or 1 to 1 correspondence between the set X and set Y.

So if I take this example here set X and set Y I have the same cardinality because they have the same number of elements namely 4. But if I see it in terms of a function then I can say that the set X and set Y have the same cardinality. Because there is a bijection between the set X and the set Y.

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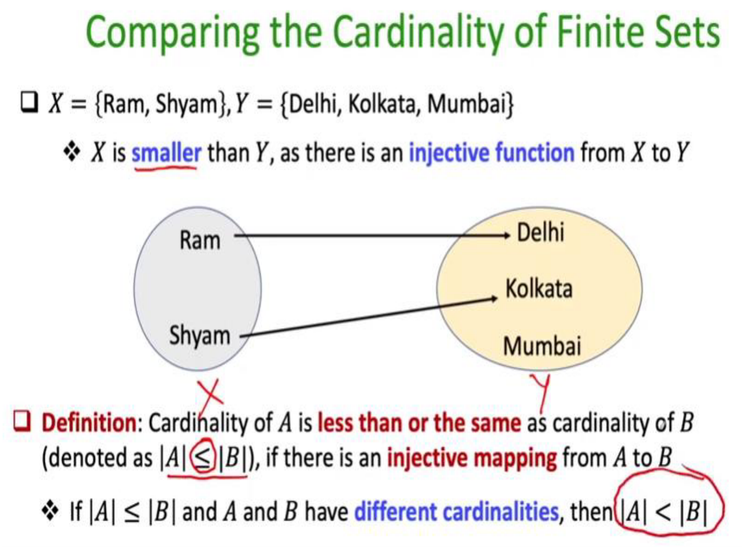

Now how do we compare the cardinality of finite sets; say if I am given the set X and the set Y. It is easy to see that the set X its cardinality is less than cardinality of set Y. Namely the number of elements in the set X is less than the number of elements in the set Y. That is why the cardinality of X is less than the cardinality of Y. But if I view the same thing in terms of a function I can say that the set X is smaller than the set Y in terms of size because there is an injective function from the set X to the set Y.

So there can be many injective functions one possible objective mapping I can define as Ram getting mapped to Delhi and Shyam getting mapped to Kolkata right. That means you take any element from the set X it will have an image a distinct image from the set Y. And distinct elements of the set X will have distinct images so that automatically ensures that the number of elements in the set Y as to be at least as large as the number of elements in the set X.

So based on this example I can give the following definition I can say that the cardinality of the set A is less than or the same as the cardinality of set B. And we use this notation the cardinality of A less than equal to cardinality of B provided there is an injective mapping from the set A to B. You might be wondering why I am using less than equal to notation here because there might be more elements in the set B compared to the set A.

That is why the less than equal to notation. Now if the set A cardinality is less than equal to cardinality of set B and the cardinality of the set A and the cardinality of set B are different, then clearly it implies that the cardinality of set A is strictly less than the cardinality of the set B right.

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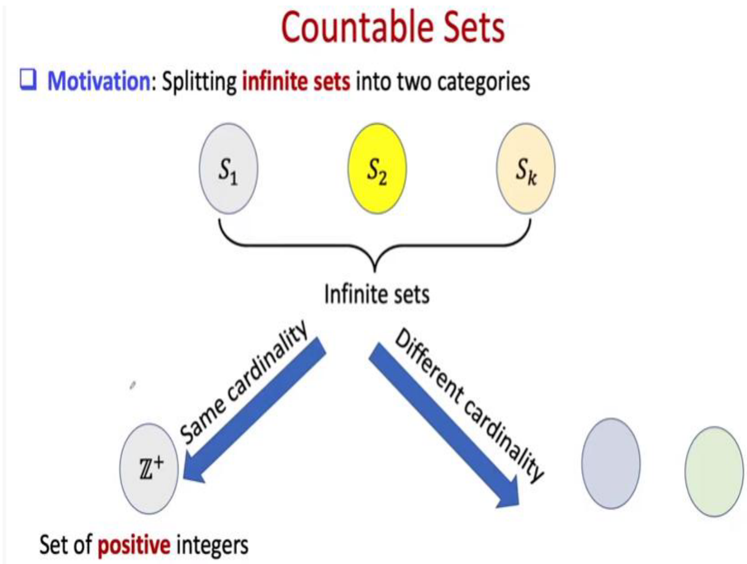

So now what are the countable sets so before going into the definition of countable sets let us see some motivation. Why we want to study countable set: the whole motivation behind countable sets is that we want to split the study of infinite sets into 2 categories. What are infinite sets: on a very high level they are sets which have infinite number of elements. So what we want to basically do is there might be several sets possible which are infinite, set of integers, sets of real numbers, set of irrational numbers and so on.

So what we want to do is we want to categorize these infinite sets into 2 categories. Sets which have the same cardinality as the set of the positive integers. And a set of positive integers are denoted by this notation $\mathbb{Z}^+$. And we want to categorize the infinite sets into other category which have different cardinality than the cardinality of the set of positive integers. That is the whole motivation of defining this notion of countable sets.

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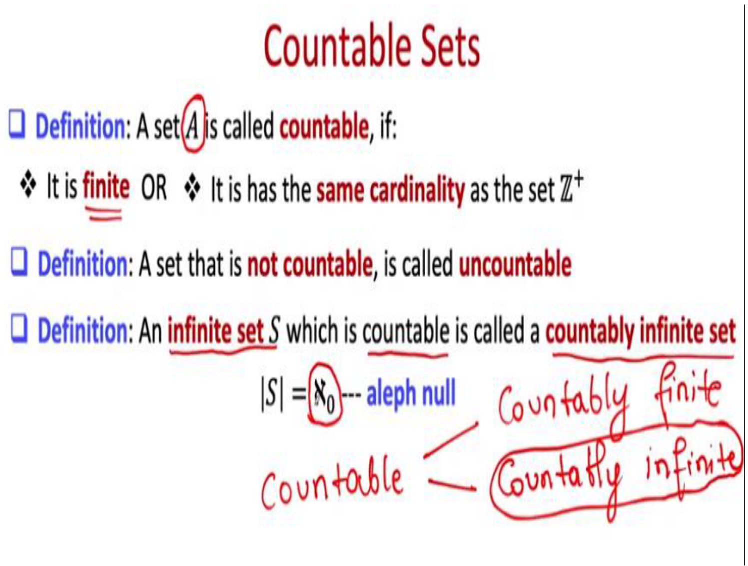

So when exactly we say a set A is countable? We say a set A is countable if it satisfies one of the following two conditions either it has to be finite namely it has finite number of elements or it has the same cardinality as the set of non-negative integers, namely the set of positive integers to be more precise. It has to be the same cardinality as the set of positive integers. If one of these 2 conditions are satisfied then we say that the set A is countable.

Whereas if a set A satisfies neither of the condition then we will say that the set is not countable. So if you are given an infinite set say S which is countable so since the set is infinite that means definitely we cannot say how many elements the set S has. But if its cardinality is same as these set of positive integers then we will call the set S to be countably infinite.

So we have countably finite sets and we have countably infinite sets so countable sets can be categorized into 2 categories either they can be countably finite that means its cardinality is finite. Or there are infinite number of elements in the set but they are countable in the sense that their cardinality is same as the cardinality of set of positive integers. Whereas if the set is not countable then we will call it uncountable.

So if we are considering the second category of countable sets namely infinite sets whose cardinality is same as the set of positive integers then we use this notation aleph null ($\aleph_0$) to denote the cardinality of such sets. So this is a Hebrew character the aleph character and 0 denotes the null feature here. So the cardinality or the size of the set of the positive integers is called as aleph null and if your set if countably infinite then its cardinality will be $\aleph_0$.

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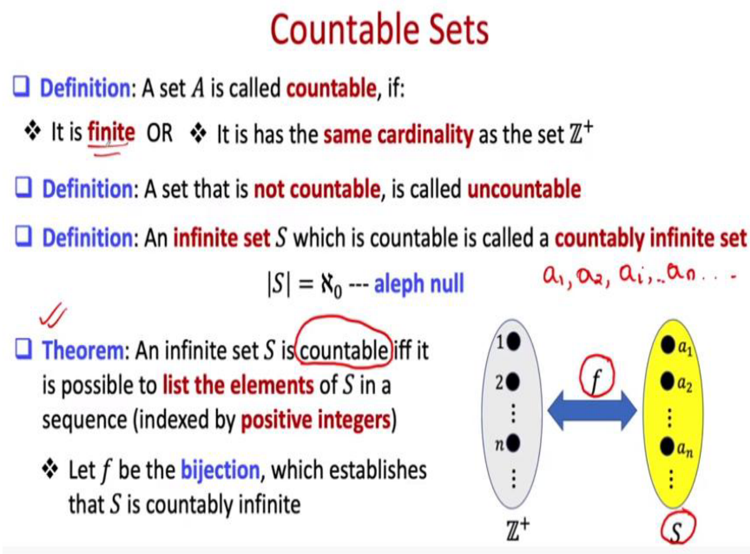

So now we can quickly prove this theorem which will be very useful later on. If you are given a set S which is countable that means your set S is countably infinite. Then it is countable if and only; if it is possible to list the elements of the set S in the sequence indexed by positive integers. So what it basically says is that even if you have infinite number of elements in the set S it is countable in the sense I can give you the sequence, I can give you a method a well-defined method according to which you can list down the elements of the set S.

And the method is well-defined in the sense that no elements of the set S will be missing as per the definition of that sequencing. That means the way I will give the definition of the sequencing or the listing of the element; each element of the set S will appear somewhere in that sequence. In that way it is a well-defined sequence and no element will be repeated in that sequence. So in that sense only my set is countable even though it has an infinite number of elements.

So the proof of the theorem is very simple since your set S is countably infinite that means its cardinality is same as set of positive integers and the set of positive integers are 1, 2 upto infinity and there has to be a bijection; call the bijection as f between the set of positive integers and your given set S. Now if in that bijection the element a_1 is mapped to integer 1 and element a_2 is mapped to integer 2 and so on. Then what I can say is that a_1, a_2, a_i, a_n is a valid sequencing or valid listing of the elements of the set S.

Why it is valid because first of all no element in this sequence is repeated and if you take any element in the set S it will appear somewhere in this sequence; it is not the case that no element will be missed in this sequence. Even though there are in finite number of elements. So what this theorem basically says is this: That if you want to show a given set to be countable and there are two ways to do that if it is finite just find out the cardinality of that set.

But if it is infinite then the only way you can show it is countable is either show a bijection or you give me a well-defined sequence or well-defined way of sequencing in the elements of that. That is equivalent to saying that you are giving me a bijection between the set of positive integers and the given set.

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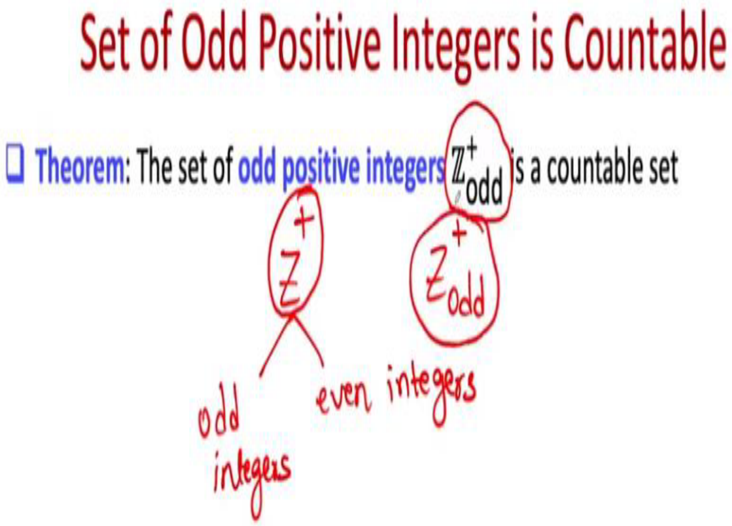

So now based on these definitions let us see some examples of countably finite sets. So I start with the set of odd positive integers and let me denote the set of odd positive integers by this notation $\mathbb{Z}_{odd}^{+}$. And my claim is that the set of what positive integers is the countable set. So the statement might look very non-intuitive because definitely you have more integers than the set of odd positive integers right.

The set of integers; if I consider the set of positive integers to be more precise then it has both odd integers as well as even integers. And both these sets; the set of odd positive integers and the set of even positive integers each of them is an infinite set. What I am going to show here is that cardinality wise the set of positive integers and the set of odd positive integers - their cardinalities are same by showing a bijection.

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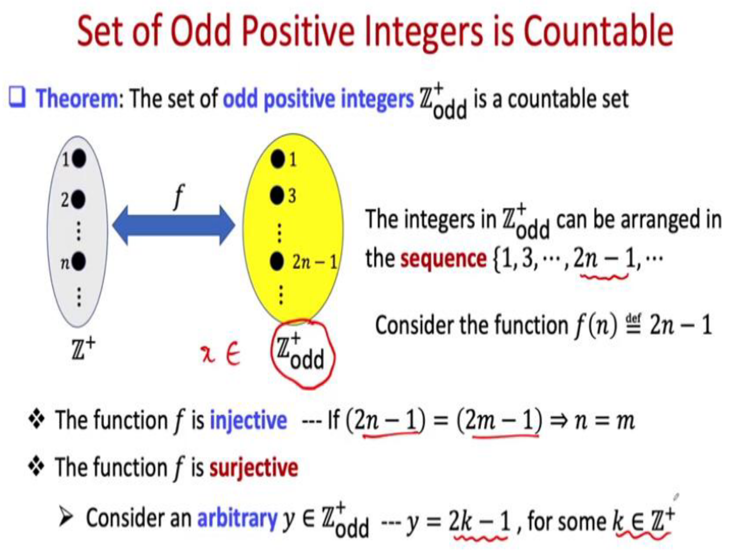

So the bijection is as follows. So consider the sequence 1, 3, 5, 7 like that where the nth term is the sequence is 2n – 1 and so on. So clearly this sequence is sequence of infinite positive integers and each number in this sequence is odd. My claim is that this is a valid sequence for the set of odd positive integers it is valid in the sense no element in this sequence is repeated. And you take any integer from the set of positive integers it will appear somewhere in this list.

It will not be the case that you keep on traversing the sequence infinitely but still you never come to that element which you are considering. It give me any element X belonging to the set of all positive integer it will appear somewhere in this sequence. To be more precise if you want to see the exact bijection between the set of off positive integers and the set of positive integers. You consider the mapping f(n) where f(n) is 2n-1.

And it is very easy to see that the function is injective because if you take the integers 2n-1 and 2m-1 and if they are same then that is possible only if your n is equal to m and clearly the function is surjective as well. How? you take any element y belonging to the set of odd positive integers corresponding to that element y, since that element y is an odd positive integer it can be written in the form of some 2k- 1 for some k in the set of positive integers.

So the pre-image of y will be the element k so that is a bijection and that shows that cardinality wise the set of odd positive integers is same as the set of positive integers. Even though, intuitively you have more elements in the set of positive integers than the set of odd positive integers.

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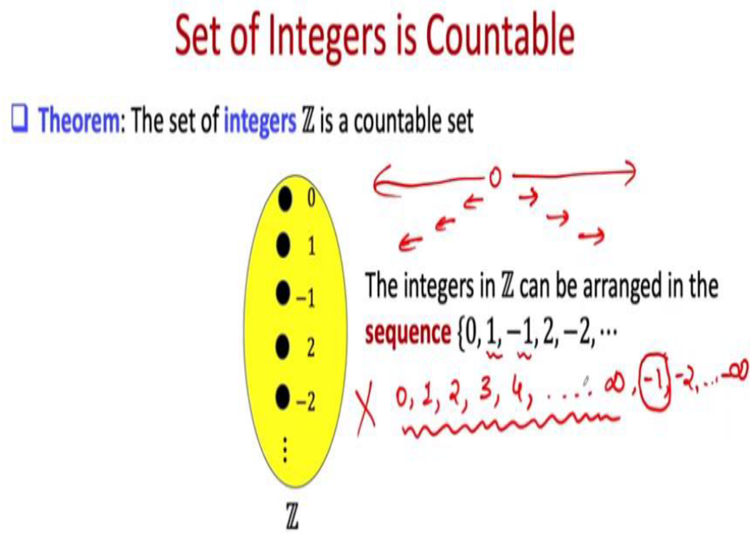

Now what we are going to show next is a very interesting fact we are going to show here that the set of integers is a countable set. Mind it the set of integers has both positive integers as well as negative integers. But what we are now going to show is that cardinality wise the set of integers is same as the set of odd positive integer, they have the same number of elements. So again I will do it by giving you a valid sequence and also by giving you an explicit bijection.

So let me first how you a sequence. My sequence here is the following I start with 0 and then I alternatively list down positive numbers, negative numbers. And then I gradually go along the number line both in the positive direction as well as in the negative direction. My claim is that this is a valid sequence in the sense you take any integer either it will be positive or it will be negative and it will appear somewhere in the way I am listing down the elements here.

Let me ask you a question here, you can think in your mind. If I consider a sequence where I first try to write down all the positive integers. And then followed by the negative integers can I consider this to be a valid sequence for listing down the set of integers. The answer is no this is not a valid sequence why this is not a valid sequence? Because if you take a negative number say -1 you do not know where exactly -1 is going to appear in this list.

Because you are starting with the set of positive integers and when you are actually traversing the set of positive integers you are going to an infinite limit. There is no guarantee that you will come back and then start enumerating your negative integers. In that sense this enumeration is an invalid from the enumeration or invalid listing. Whereas the way I have listed down the elements where I am alternately listing down the positive negative integers.

That means along the number line I started with a 0 and then in each step I am listing down the next positive number and then the next negative number. Then the next positive number then the next negative number then the next positive number then the next negative number. Now if I consider this sequencing, even though your given integer which you want to find out where exactly it is appearing in the sequence is very large.

It might be appearing somewhere; you do now know where exactly it is appearing. It will eventually be enumerated in the sequence and you will never miss it. That is why it is a valid sequencing, it is a valid listing. Whereas the listing that I have considered here where I am first trying to enumerate all the positive integers followed by all the negative integers it is not a valid sequence.

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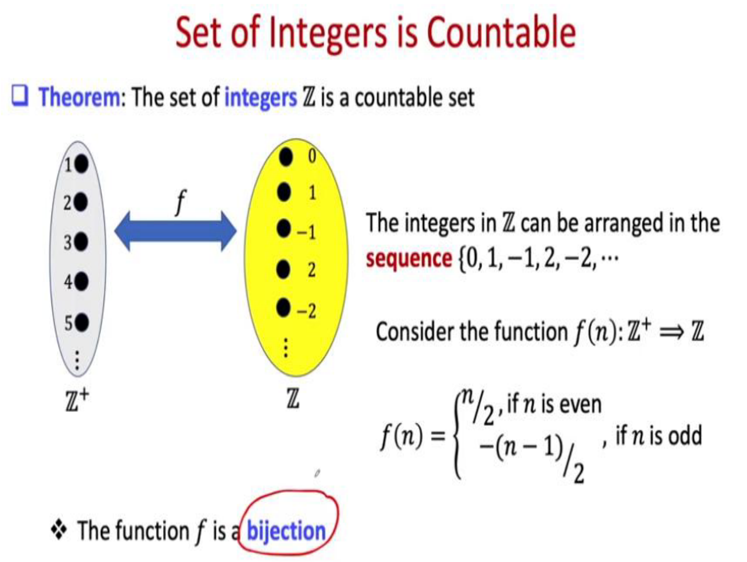

Now if you want to see an explicit bijection between the set of integers and the set of positive integers consider the following mapping. My f(n) will be n / 2 provided n is even whereas if n is odd that means (n – 1) is divisible by 2 then my mapping will be –(n – 1) / 2. And it is the straight forward exercise for you to verify that indeed is mapping f is a bijection which shows that the cardinality of the set of positive integers and the cardinality of integers are the same.

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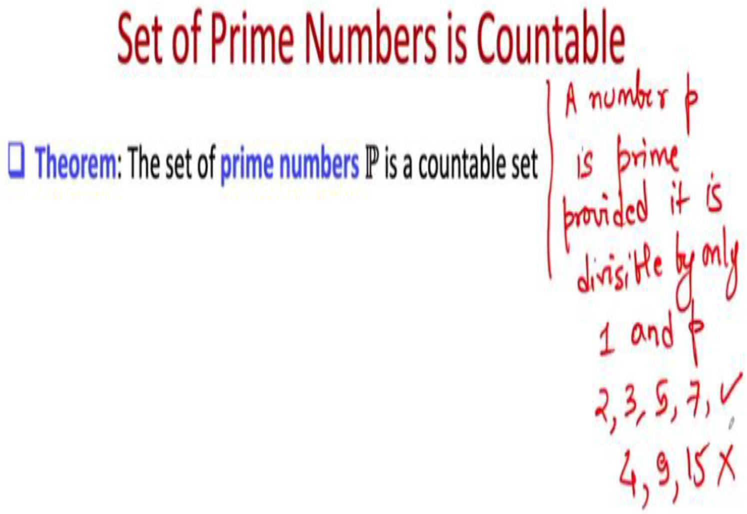

Next let us consider the set of prime numbers and for those who do not know what is a prime number? A number p is prime provided it is divisible by only 1 and p; that means there are no other divisors for the number p apart from the number 1 and the number p. Of course 1 divides any number. And the number divides itself by default these are the 2 valid divisors of any number.

If your number p is such that you do not have any other divisors other than the number itself and 1 then the number p will be called as the prime number. So if you take say 2, 3, 5, 7 they are all prime numbers. Whereas if you take the numbers like 4, 9, 15 they are not prime numbers because divisors of 4 are 2, divisors of 9 are 3, divisors of 15 are 3 and 5 and so on. So my claim is that a set of prime numbers is a countable set.

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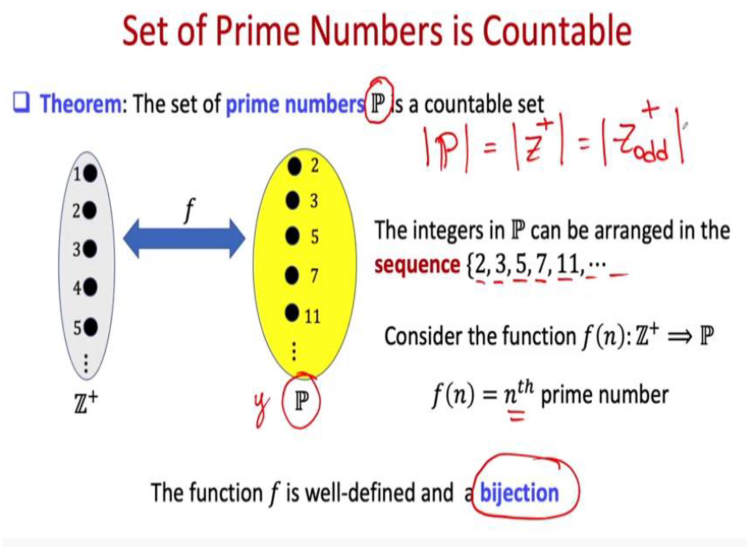

So again those who are not familiar with number theory they might be knowing that there are infinite numbers of prime numbers. But what we are starting in this theorem is that cardinality wise the number of prime numbers, the cardinality of the set of prime numbers is same as the cardinality of the set of positive integers. So this I can either prove by giving you a valid sequence and the sequence is very straight forward.

You just enumerate the prime numbers in increasing order that is all and it is a valid sequence because this is an infinite sequence; that is fine. But it is valid in the sense that every number in this sequence is a prime number, no number is going to be repeated and you take any element y belonging to the set of prime numbers it will eventually appear in this sequence. It will not be the case that you do not know where exactly; it is not the case that this number y will never appear in this sequence.

Even if you are traversing towards infinity, if you are interested in the explicit bijection between the set of positive integers and the set of prime numbers, the bijection is very straight forward. You just output the nth prime number as the value of your function f on input n. And the function is well defined and clearly it is bijection. So now what we have proved till now? We have proved will now that the set of prime number, it has the same number of elements as the set of odd positive integers which has the same number of elements as the set of positive integers. That means even though element wise they are different but cardinality wise they are same.

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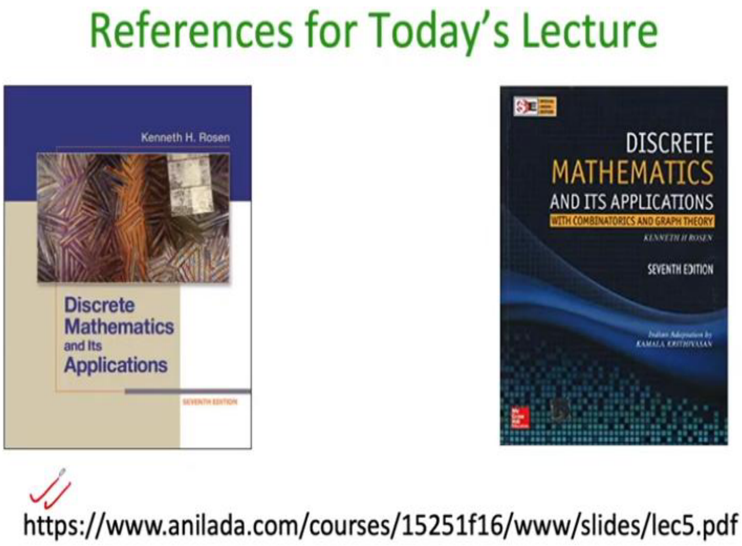

So that brings me to the end of this lecture. These are the references for today's lecture. We are of course following the relevant chapters from Rosan book. But this is another interesting document which you can refer where you have a very nice explanation regarding the whole cardinality theory, the theory of countable sets and just theory of uncountable set. So that brings me to the end of this lecture, thank you.