Discrete Mathematics
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Lecture -15
Sets

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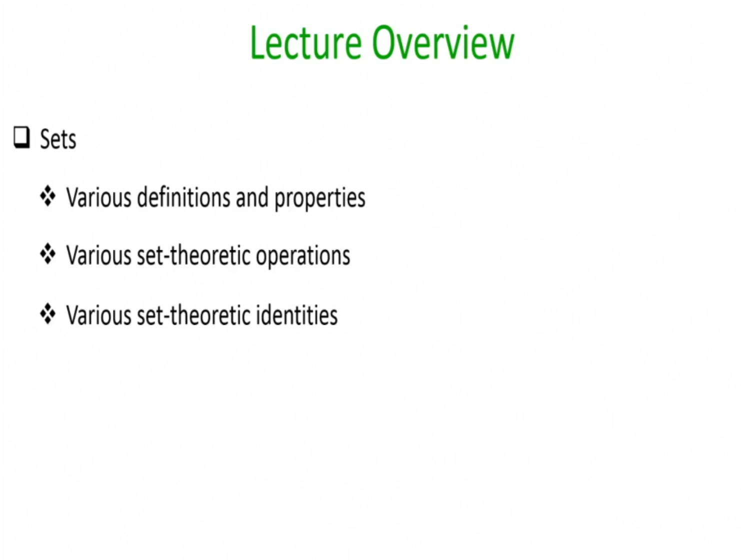

Hello everyone, welcome to this lecture on sets. The plan for this lecture is this follows, we will introduce the definition of sets, we will introduce various set theoretic operations and we will discuss various set theoretic identities.

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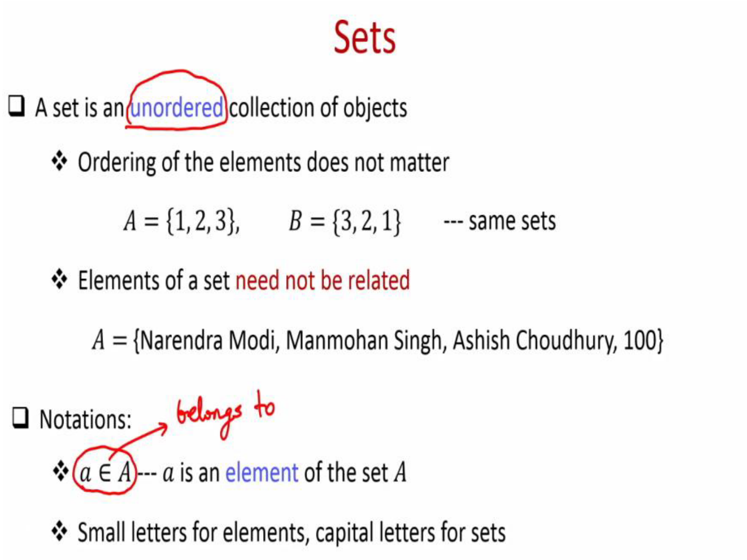

So, what is the definition of a set? A very high level definition is, it is an unordered collection of objects and what I mean by unordered collection of objects here is that ordering of the elements in the set does not matter. So, for instance, if I have a set consisting of the elements 1, 2 and 3 and then it does not matter whether I list them as 1, 2, 3 or whether I list them as 3, 2, 1 both will be the same sets.

It turns out that the elements of the set need not be related. So, for instance, if I have a set consisting of entities, Narendra Modi, Manmohan Singh, Ashish Choudhury and 100 it is a valid set as far as the definition of a set, because the definition does not say anything regarding the properties of the elements of the same. We use some well known well-defined notations for representing sets.

So, we use this notation $\in$ for a belongs to A. So, this notation "belongs to", whenever a is an element of set A we use this notation. And throughout this course, we will follow the notation that we will be using small letters for elements of the sets, and we will be using capital letters for the sets.

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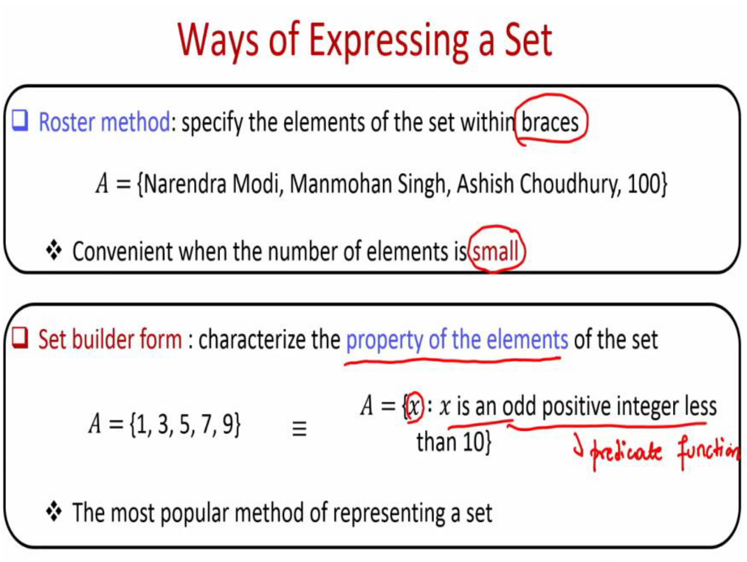

Now, how do we express a set? There are two well known methods. The first method is the Roster method, where we specify the elements of the set within braces. So, for instance if A is a set consisting of 4 elements, then I have listed down the elements of the set A and this is a convenient way of representing a set provided the number of elements in the set is small. If the number of elements in the set is extremely large then it will not be feasible to write down or list down all the elements of the set explicitly.

So, that is why we use the second form or second way of expressing a set which is also called as the set builder form and what we do here is that instead of listing down the elements of the set, we write down or state the general property of the elements of the set, which is specifically specified by a predicate function. So, for instance here is a set A consisting of elements 1, 3, 5, 7, 9 in the Roster method.

The same set can be expressed in the set builder form where I can specify that A is the collection of all odd positive integers x which are less than 10. That means I am basically stating the properties of all the elements of the set A. So, I do not need to explicitly write down all the candidate values of x satisfying this property. I am just specifying the general property and you can imagine that this general property is a predicate function.

So, you can imagine that in the set builder form we specify the predicate function, which is applicable on all the elements of the set and this is the most popular method of representing a set specifically if we are dealing with an infinite set.

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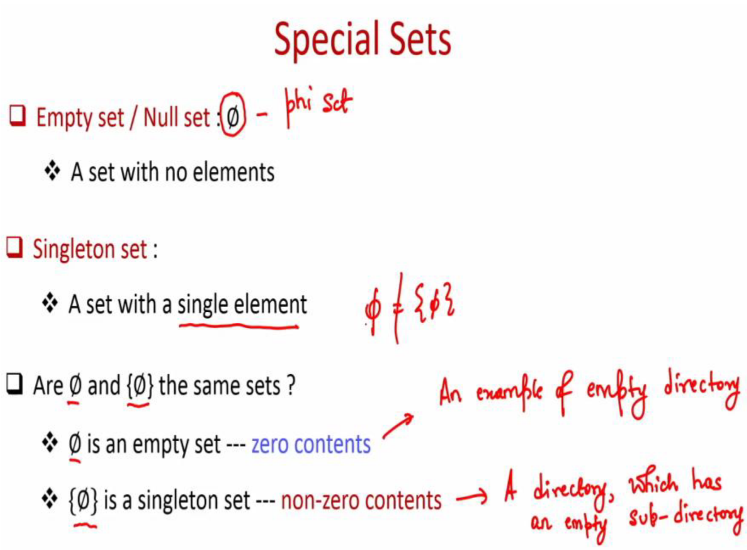

We often encounter some special sets. So, a null set or the empty set is one of them and this is the notation ϕ which we use to represent the null set. This is also called as phi set or phi set and it is a set which has no elements. So, you can imagine that a directory which has no files inside it is an example of an empty set. Another special set which we encounter is the singleton set and it is a set which has a single element in it.

Now an interesting question is that are these two sets the same? So, I have the set ϕ and I have a set which has an element ϕ and it turns out that these two are different sets. If I consider the set ϕ, then it is a set which has zero content, it has no element in it. Whereas if I consider the set specified by this notation $\{ \phi \}$ namely we have the braces, within the braces we have this ϕ and this is a singleton set because it has one element and hence it has non-zero content.

So, analogy here is you can imagine that ϕ is an example of an empty directory, which has no files inside it. Whereas this notation $\{ \phi \}$ namely the set specified by this parenthesis within which you have this ϕ can be interpreted as it is directory which has a sub directory or which has specifically an empty subdirectory within it and clearly these two things are different. So, very often people get confused.

They think that the set ϕ is equal to the singleton set consisting of element ϕ that is not correct they are two different sets as soon as I put a parenthesis around ϕ the meaning completely changes.

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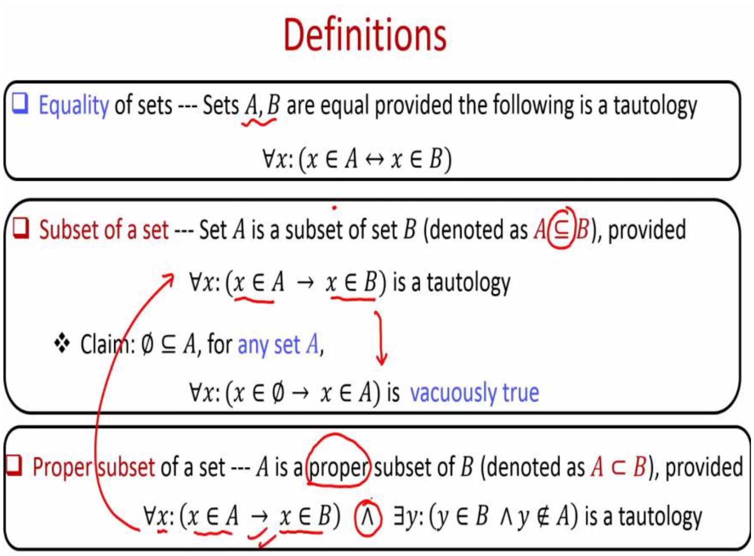

So, now we will introduce some definitions in the context of sets. So, we start with what we call as equality of sets. So, intuitively if I have two sets A and B they will be called, or they will be considered equal sets if they have the same elements. That means if I have any element present in the set A it is present in B and in the same way any element which is present in B is also present in A there is nothing which is extra present in A or which is extra present in B.

So, this is formally stated by the following definition. We say that the sets A and B are equal provided the following statement is a tautology namely for all x, of course the domain of x here is the set of elements in A and B which is not explicitly specified here. So that expression says you take any x from the domain if it is present in A then it should be present in B and vice versa because this is a bi-implication.

And this bi-implication will be tautology only when if left hand side is true right hand side should be true and vice versa. The next definition is a subset of a set. So, if I have two sets A and

B then the set A is called a subset of the set B and for denoting that we use this notation ($\subseteq$), provided the following holds, you take any element in the set A it should be present in B that means it should not happen that there is something in A which is not there in B.

This is stated formally by saying that the following expression should be a tautology namely for all x in the domain, if x is in A then it should be present in B. I do not care what happens if x is not present in A. I do not care for those elements x. I am interested only for those elements x which are present in A. My requirement is they should be present in B. So, my claim is that the empty set is always a subset of any set does not matter whether the set A is empty or not.

The empty set is always a subset of any set and if you are wondering why this is the case then you apply this definition on the set ϕ, then ϕ will be a subset of A. If the following implication is a tautology namely for all x, if x is an element of ϕ it should be an element of A. And it turns out that this implication is vacuously true because what is the premise here, the premise here is that element x belongs to ϕ.

But that is false because ϕ is an empty set and x belonging to ϕ is defined to be a false statement. It is a false statement because ϕ does not have any content. So, this statement is vacuously true. This implication is vacuously true and that is why ϕ is always a subset of any set. You also have what we call as proper subset of a set. So, A will be called a proper subset of B and for this we use this notation ($\subset$).

So, you see here that this equal to symbol which was present in the notation for subset is missing in this notation. So, we will say A is proper subset of B provided there exist at least one element in B which is not in A. So, we still have the requirement at everything in A should be present in B, but it might be possible that A is equal to B. If A is equal to B then in that case also we will be saying A is a subset of B.

But when I say proper subset, by the word proper here, I mean that there is something extra in B, which is not there any A. So, more formally the following expression should be a tautology for every x in the domain if x is present in A it should definitely be present in B. So, this is the requirement of subset, this captures the fact that A is a subset of B plus I need something extra.

That is why conjunction and what is the extra thing here, there should be some element y in the domain which should be present in B, but it should be absent in A. At least one such y should be there. If no such y is there but only this part is true, then the proper subset definition turns out to be the same as that of a subset.

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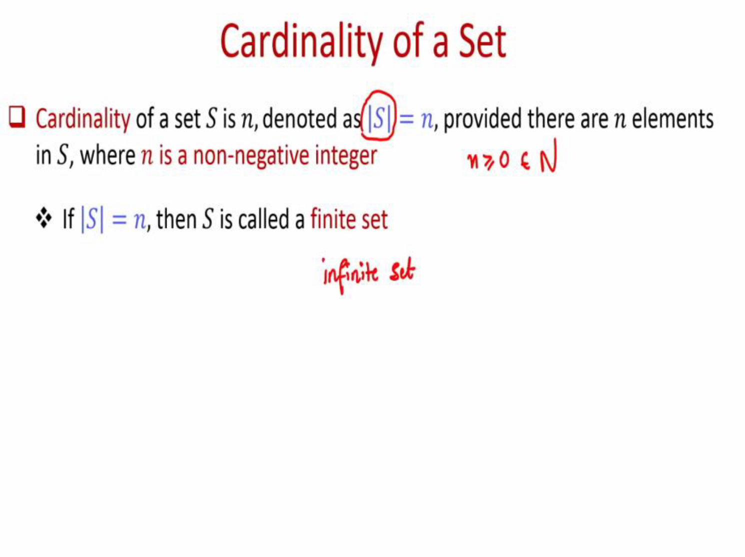

Next we define the cardinality of a set. So, we say that cardinality of a set S is n and for that we use this notation. We use this to vertical bar symbols (| |) within S to denote its cardinality and we write it is equal to n provided there are n elements in S where n is some non-negative integer. So, n could be 0 or 1 or it should be some value belonging to the set of natural numbers or it should be a non-negative integer.

So, if the cardinality is some n where n is a non-negative integer then we say that S is a finite set else we say it is an infinite set. That means if we cannot express the number of elements in a set by any non-negative integer then the cardinality of the set will be considered as infinite.

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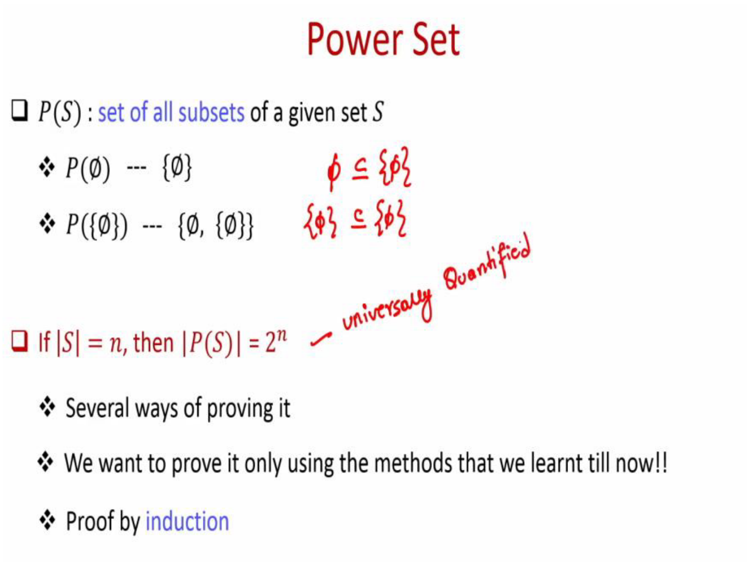

We next define what we call as the power set of a set and we use this notation P(S). So, you are given a set S. And if I take the collection of all subsets of this set S, then that itself is a set because I am just listing down the subsets of S and the elements here the elements of P(S) are the subsets of S. So, if I list down all the subsets of S the resultant set is called as the power set. So, let us try to find out that what will be the power set of ϕ.

So, it turns out that a power set of ϕ will be a singleton set consisting of the empty set, because empty set is always a subset of itself or any set. What will be the power set of this singleton set which has ϕ as its element, it turns out that the power set will have two elements because ϕ is a subset of any set. So, we have ϕ subset of a singleton set consisting of ϕ and the singleton set consisting of ϕ by default is always a subset of itself.

So, we have two subsets of the singleton set consisting of the element ϕ. Now a very fundamental fact here is that, if the cardinality of your set is n where n is some non-negative integer, then the cardinality of the power set will be 2^n. There are 2 to the power n possible subsets of a set consisting of n elements and this is a very interesting fundamental fact which can be proved in several ways.

But what we will do now is we will try to prove it using the methods that we have learnt till now, namely the proof mechanisms that we have seen till now. And since this is a universally quantified statement, applicable for all n, a natural choice here to apply the proof by induction namely we will prove the statement by induction on the value of n.

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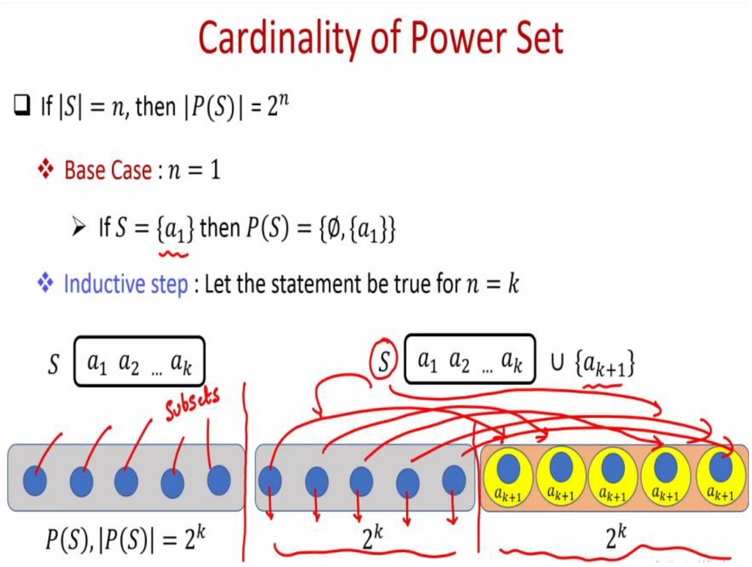

So, my base case will be n equal to 0, we start with base case n equal to 1. Of course, you can start with base case of n equal to 0. If n is equal to 0 that means your set is the empty set and hence its power set will have only one element. But I start with base case n equal to 1. So it is easy to see that if my set has only one element then the power set will have two elements. Namely there are two subsets of this singleton set the set ϕ and the set itself.

Now let us prove the inductive step and for that let me assume that the inductive hypothesis is true. Namely I assume that the statement is true for any set consisting of k elements that means I am assuming that if my set S has elements a_1 to a_k, then the number of subsets of this set S is 2^k. So, all these blue circles are the subsets of S and there are 2^k such circles, that is my inductive hypothesis.

Now I am increasing the number of elements in my set S by 1 more, I am adding 1 extra element. A new element, which is not there in the existing set and now I have to find out the number of subsets of this new set S. So, it turns out that all the old subsets of the old set S are going to be subsets of this new set S. Because if I do not include a_{k+1} in a subset of S, then that is still a valid subset of S.

That means whatever sets for the subsets of old S they are still the subsets of new S and how many such old subsets I have, I have 2^k subsets because that is coming from inductive hypothesis. And now what about subsets of this new S, which has the element a_{k+1}. It turns out that those new subsets I can form by taking these old subsets and adding the new element a_{k+1} to it.

There is a 1 to 1 correspondence here. This is because in each of this old subsets the element a_{k+1} was not present and if I add a_{k+1} to that, that becomes a valid subset of S, and all these new subsets, which I have constructed they have not been counted earlier and how many new subsets I can form? It is exactly the same as the number of subsets of old S which is 2^k. And it turns out that any subset of S can be either of this type or of this type.

Namely it either will have the element a_{k+1} or it will not have the element a_{k+1}. If it does not have the element of a_{k+1}, that means those subsets are of the type this. Whereas the a_{k+1} is present, then those subsets are of type this. So, overall, how many subsets I obtain? $2^k + 2^k$ which is 2^{k+1} and that proves my inductive step.

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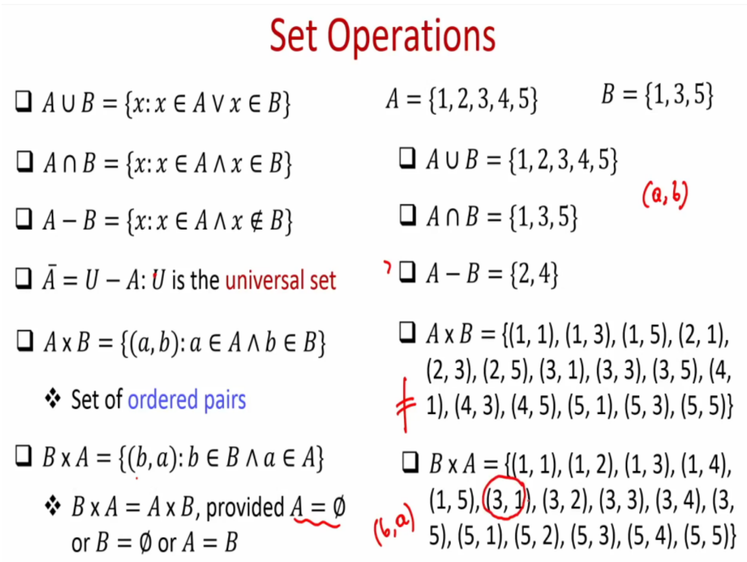

So, now let me introduce some set operations and most of you will be familiar with this. This is the union (∪) of two sets and it is the collection of all elements x in the domain which are present in either A or present in B. Of course, it might be present in both of them because this disjunction will be true if the condition x belongs to A is simultaneously true as well as condition x belongs to B is also simultaneously true.

Then we have the intersection of two sets ($\cap$) and it consists of all the elements x in the domain which are present in both A as well as in B. That is why we have a conjunction here. That is, both the conditions x belonging to A and x belonging to B should be true. These are two fundamental operations. Then we have the set difference is A - B is called as the set difference and A - B consist of all elements from the domain which are present in A but not in B.

And 'but' is represented by conjunction. Then we have this operation called A complement ($\bar{A}$). And A complement is defined with respect to a universal set which you can imagine as kind of a bigger set. So, if you subtract a set A from the universal set whatever is left, it is called as A complement, denoted by this notation A bar ($\overline{A}$). Now this is an important operation A cross B which is called as the Cartesian product of A and B.

And what exactly this set is, well it consists of all ordered pairs of the form (a, b). Where the first component of the ordered pair should be from the set A and the second component of the ordered pair should be from the set B. So, if I collect all such ordered pairs then the collection of those ordered pairs will becomes called A x B. It is important to note here that the order matters a lot here when I take the Cartesian product when I am saying A x B, then the first component of the ordered pairs should be from the set A and the second component should be from the set B whereas if I take the Cartesian product of B and A then it will be collection of all ordered pairs with the first component in the set B and the second component from the set A. So, it is not always the case that A x B is equal to B x A that can happen only in some special cases.

Namely we can very easily show that the Cartesian product of A and B and the Cartesian product of B and A are same provided one of the following three holds : either your A should be an empty set because if A is the empty set then it does not matter what is B, A x B will always be empty and B x A also will be empty. Same holds if B is equal to empty set. Whereas if A is equal to B in that case also A x B and B x A will be same.

Because, all ordered pairs of the form (a, b) will also be encountered in the Cartesian product of B x A. So, these are the only three cases when the Cartesian product of A and B will be same. Otherwise the Cartesian product of A and B need not be same. So, let me demonstrate these operations with some example here. So, I take these two sets A and B. Union means I pick all the elements which are both in A and B.

So, since 1, 3, 5 are already there in A, I do not have to separately write down 1, 3, 5 again because the definition of set says that I will be listing down the elements of the set only once even if it is appearing multiple times. The same way the intersection here, so, what are the common elements? I have 1 present in both A and B, I have 3 present in both A and B and I have 5 present in both A and B and it is a set 1, 3, 5, which will be considered as the intersection, because the elements 1 3 5 satisfies this predicate condition in the definition of $A \cap B$. What will be A - B? So, the definition of A - B means all the elements which are only in A, but not in B. That means I have to subtract out all the elements which are in B from A as well. So, 1 and 1 cancels out, 3 should not be included, 5 should not be included; that means I am left only with 2 and 4.

So, it is only the elements 2 and 4 which satisfies the definition of A - B. Now if I take the Cartesian product of A and B. That means I will be now taking all elements of the form (a, b) where a’s will be coming from the set A and b's will be coming from the set B. So, these are the elements which I have listed down in A x B whereas if I take the Cartesian product of B and A, then it will be collection of all ordered pairs of the form (b, a).

Where b comes from B and a comes from A. And now, you can check that A x B is not equal to B x A. So, for instance, you have the element (3, 1) present in B x A, but (3, 1) is not present in A x B and there are many such elements which are there in one set, but not in the other set.

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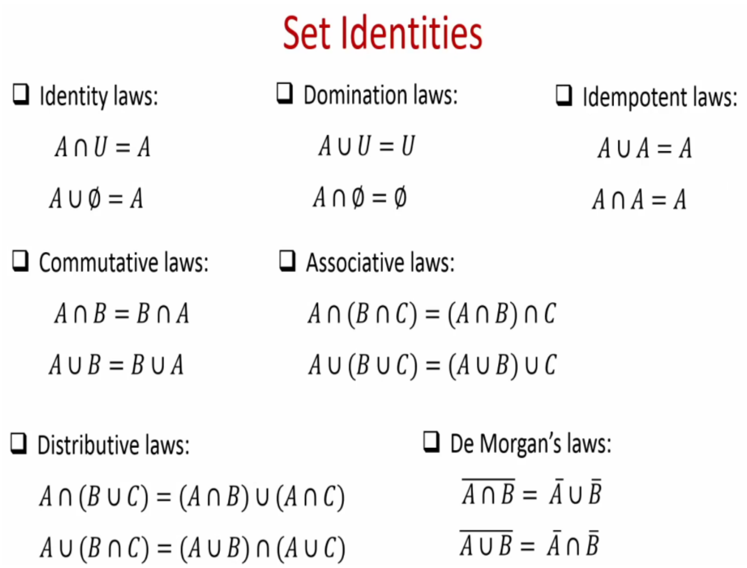

Now, there are some well known set identities which are available. We have some names also for these set identities where each of these identities basically state that a set in the left hand side and the set in the right hand side are the same. And we can prove them and assuming that these are true we have associated names with them and whenever we want to simplify expressions involving set, we can call these set identities.

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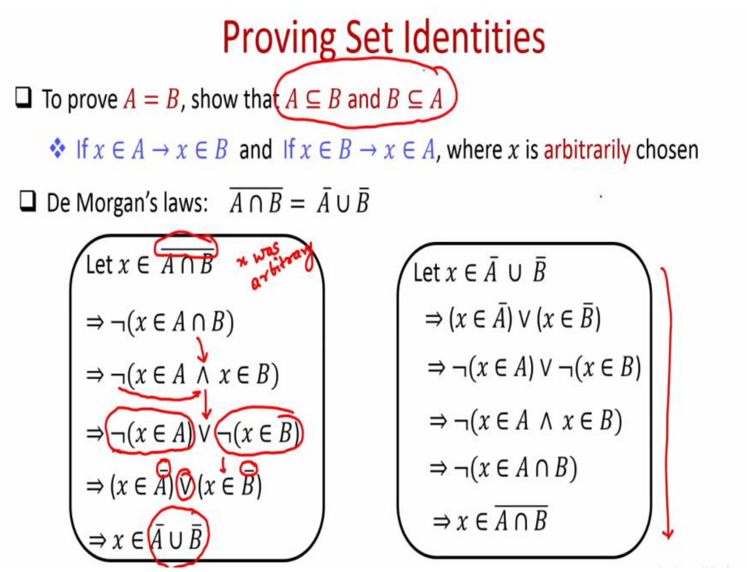

So, the question here is how do we prove a set identity, if an identity is given to us, how do I prove that the two sets A and B which are given in the left hand side and in the right hand side they are same. So, for that we have to understand here that two sets A and B are equal if they are respectively subsets of each other. Because the definition of A equal to B was for all x: x implies x belonging to A bi-implication x belonging to B.

Now, if I split this bi-implication, this means if x belongs to A it should belongs to B as well, and bi-implication can be splitted into conjunction of two implications. Now this condition means of course everything is with respect to for all x, this condition means A is a subset of B and this condition means B is a subset of A. That means to show that two sets A and B are equal, I have to show that A is a subset of B and B is a subset of A.

Namely I have to prove two implications. And the two implications are I will be taking some arbitrary element x and assuming if it is present in A I have to show it is present in B. Why I am taking x to be arbitrary here? Because remember I want to prove this universal quantification. So, since this statement has to be proved for every x in the domain. I cannot take every x in the domain and prove this implication to be true.

And that is why I apply the universal generalization here. Where the, universal generalization says that to prove this universal quantification, you prove the universal quantified statement to be true for some arbitrary x in the domain. If you prove it to be true for arbitrary x in the domain you can conclude it is true for every x in the domain. So, that is why I am taking my x here arbitrary. So, to prove that A and B are subsets of each other these are the two implications I have to show.

So, let me demonstrate what I said with respect to this example. I want to prove the De Morgan's law. The De Morgan's law is there are two variants of De Morgan's law. I am proving one of them. It says that if you take the complement of intersection of A and B that is same as the union of $\overline{A}$ and $\overline{B}$, of course, here everything is with respect to some universal set, everything is with respect to some universal set U.

Because whenever a compliment is coming into picture, we have to assume that there is some universal set U. So, I have to prove that everything here is also present in here. And everything in the right hand side set is also present in the left hand side set. So, these are the two things you have to prove. So, let me prove that everything in the left hand side set is also present in the right hand side set. So, let x be some arbitrary element present in the $\overline{A \cap B}$.

Since the x is present in the $\overline{A \cap B}$, that means it is not present in the $A \cap B$ because that is a definition of this complement operation. Whatever was present in $A \cap B$, if I separate it out that will give me the compliment; that means I can say that for those x which are present in the complement of $A \cap B$ this condition holds. The negation of this condition holds.

Now what I can do is I can apply the definition of intersection. Since x is present in $A \cap B$, that means x is present in both A as well as in B. And now I can apply the De Morgan's law of predicate logic. So, what I can do is I can take the negation inside and this conjunction gets converted into disjunction. And negation gets splitted over the individual expressions.

But if I see here the negation of this statement namely negation of x belonging to A means x belongs to $\overline{A}$ because that is coming from the definition of $\overline{A}$. In the same way this thing, that negation of x belonging to B means, x belongs to $\overline{B}$ because that is the definition of $\overline{B}$. And now, I apply the definition of disjunction here.

If x is present in $\overline{A}$ or if x is present in $\overline{B}$, that means it is present in the $\overline{A} \cup \overline{B}$. And throughout my x was arbitrary, that means I have shown that you take any member of the set $\overline{A \cap B}$, it will be present in the set $\overline{A} \cup \overline{B}$. And now I can show the other way around as well.

I take an arbitrary element x present in the $\overline{A} \cup \overline{B}$ and by applying simple rules of logic here and using definition of $\overline{A}$ and $\overline{B}$ and negations. I end up with the conclusion that the element x will be also present in $\overline{A \cap B}$. So, that is how we prove the set identities. We have to show that both the left hand side set and the right hand side sets are same.

And for that I have to prove that both the sets are individually subsets of each other. So, that brings me to the end of this lecture, just to summarize in this lecture we introduce the definition of sets we introduced set theoretic operations and set theoretic identities.