Discrete Mathematics
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Lecture -24
Functions

Hello everyone. Welcome to this lecture.

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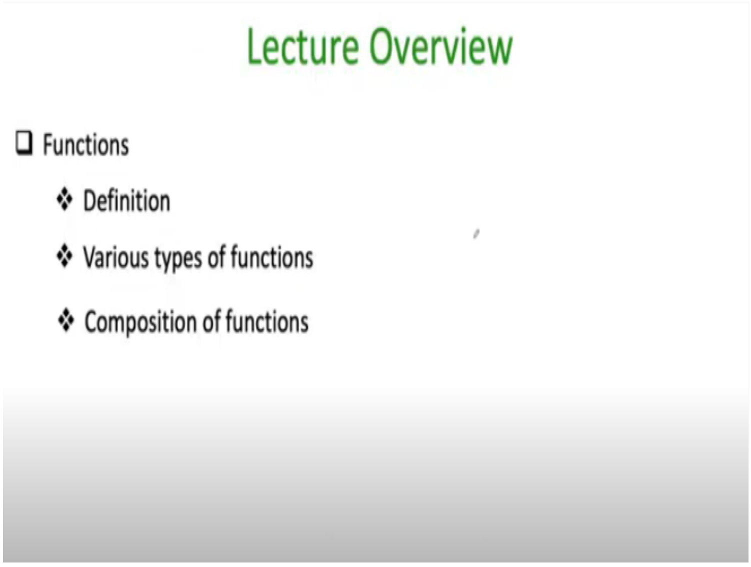

Just to begin with, just to recap the last lecture we discussed about the notion of partial ordering and we discussed about topological sort. The plan for this lecture is as follows. In this lecture, we will discuss about functions. We will see the various types of functions and we will also see composition of functions.

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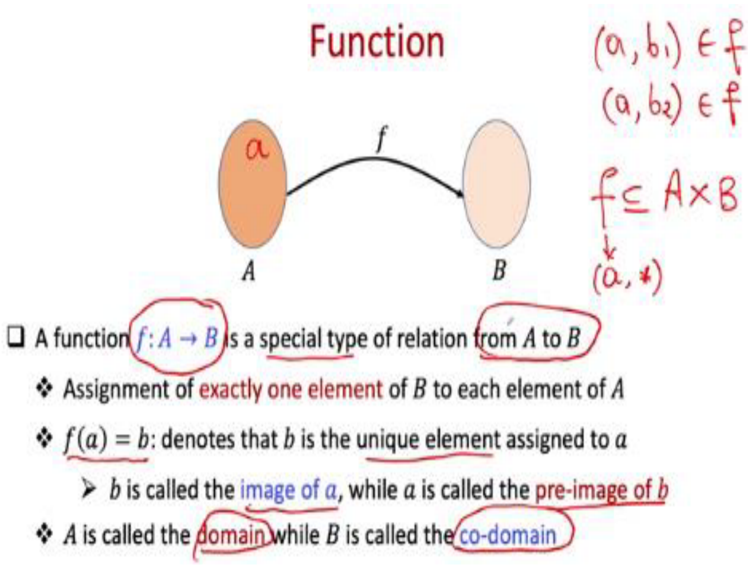

So, what is a function? So, imagine you are given two sets. Set A and a set B and when I say I have a function, say, $f: A \to B$. Then it is a special type of relation from the set A to the set B. And pictorially you can imagine that we have the set A and set B. So, they may be the same set or they might be different sets. It does not matter.

And what is the specialty about this relation? So, I am saying that the function is a special type of relation from $A \to B$. So, of course f is a subset of the Cartesian product of A and B, $f \subseteq AXB$, because that is a definition of a relation from A to B. Now, this is a special type of relation. The specialty here is that, each element of the set A is assigned exactly one element of the set B.

So, that means in terms of ordered pairs if I consider this function f, then each element a belonging to the set A will appear exactly in one of the ordered pairs in the relation corresponding to this function f. And this holds for every element a from the set A. So, none of the elements a will be missing. Each such element a will appear as part of an ordered pair in the function f. So, we use this notation $f(a) = b$ to denote that b is the unique element which is assigned to the element a as per this function f and the element b is called as the image of element a. And a will be called as the pre image of the element b.

So, again to state the specialty of this relation each element from the set A will appear as part of the as part of an ordered pair as an ordered pair in the function f. And it would not happen that you have element a mapping element a getting map to say element b_1 in the function as well as element a getting mapped to the element b_2 which is different from b_1 also in the function f.

That is possible in a relation. In a relation the same element, $a \in A$ can be mapped to multiple elements. It can be related to multiple elements from the set B. But the specialty of the function is that there will be only one element from the set B which will be related to the element. And now element from the set A will be missing as part of this function and that means every element will have a corresponding related element from the set B as part of that function.

We also use the term domain and the co domain and the context of a function. So, the set A will be called as the domain of the function while B is called as the co-domain of the function. So, notice that all these definitions are with respect to a function $f: A \rightarrow B$. The direction of the function matters a lot. If I say my function is from the set B to the set A. Then it will be it should be interpreted as a special type of relation from the set B to the set A where the ordered pairs will have the first component from the set B. And the second components of those ordered pair will be from set A. And then the domain will be the set B and the co-domain will be the set A and so on.

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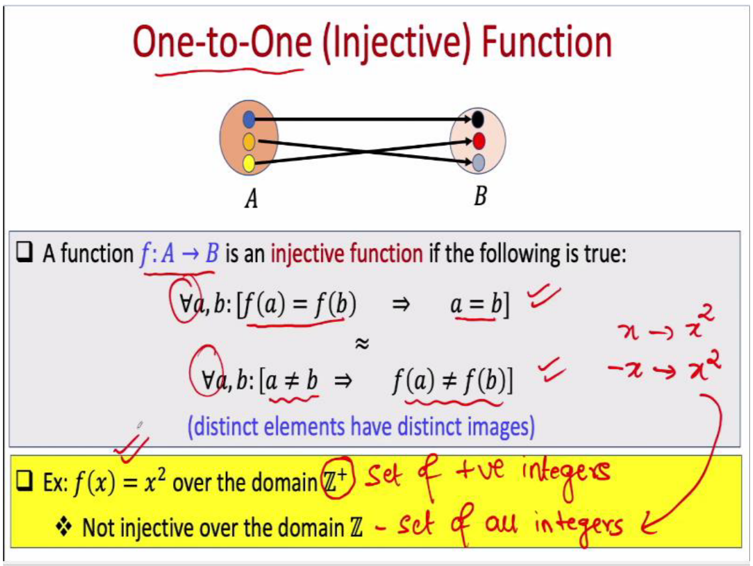

So, now we will be interested to study some important class of functions. So, the first important class of functions is the one to one or injective functions. So, imagine you are given a function f from the set A to the set B. It will be called as an injective function, provided distinct elements from the set A have distinct images. So, pictorially you can see here that each of the elements the circles from the set A is assigned a distinct element from the set B as an image.

So, to put it formally we want $\forall a, b\ f(a) = f(b) => a = b$ should hold for an injective function. So, what exactly this universal quantification means it says that if you have two elements a and b whose images are same then that is possible only if the elements themselves are the same or the pre images are the same. Or equivalently the contrapositive of the same thing is $\forall a, b, a \neq b => f(a) \neq f(b)$.

So, both these definitions are equivalent. If you prove any of these two conditions to be true for the function f and remember both of them are universal quantifications. If any of these two universal quantifications hold for your function f then we will call the function f to be an injective function. Even if there is one pair of elements here a and b for which these universal quantifications are not true then the function will not be called as an injective function.

So, you can see why it is called one to one function. The mapping is unique for every element. So, if I consider the function f(x) define to be $f(x) = x^2$ over the set $\mathbb{Z}^+$. So, this set is the set of positive integers. Then what do you think is this function is a one-to-one function? Well, it is not an injective or one to one function over the set of entire integers. If I consider the set of all integers, definitely this is not a one-to-one function because I have both $+x$ getting mapped to x^2 as well as $-x$ getting mapped to x^2, if my domain is the set $\mathbb{Z}$ if my x can take positive values as well as negative values.

But if I restrict my function only over the set of positive integers that means my $x \in \mathbb{Z}^+$ and clearly this is an injective function. So, you can see the importance of domain. If you change the domain, the interpretation or the meaning or the property of the function changes immediately.

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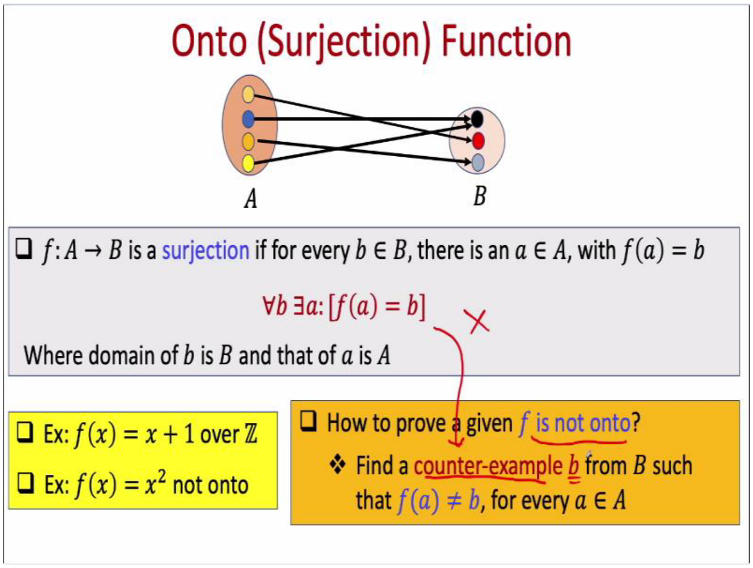

The next important category of function is the onto or surjective functions. So, again imagine you are given a function $f: A \to B$. If we call it will be called as a surjective function provided the following universal quantification hold. You take any element b from the co-domain, it should have at least one pre-image. That is the condition.

So, you can see here and the universal quantification the domain of b is the co-domain of the function. And the domain of a is the domain of the function. So, pictorially you take any element from the set B, there should be at least one pre-image for that element. It might have multiple pre images as well. So, for instance, if you take this particular element, it is the pre-image of two possible elements. It is the pre-image of this element as well as it is the pre-image of this element.

The definition simply says there should not be the case that you have some element in the co-domain set which do not have any pre-image. That will be a counter example for this universal quantification. So, even if you have any such counter example, the function will not be called as a surjective function. Whereas if this universal quantification is true for all b, a where the domain of b is the B set and the domain of a is the A set, the function will be called as a surjective function.

So, here are two functions, the first function is over the set $\mathbb{Z}$ as $f(x) = x + 1$. Now, what can you say whether it is a surjective function or not? Well, it is a surjective function. So, what you can prove is, you take any integer $y \in \mathbb{Z}$. Its pre-image will be $y - 1$. That means the element y

- 1 would have been mapped to the element y and this is true for any y from the co-domain set. So, that is why the first function is a surjective function or onto function. But the second function is not onto because $x^2 \in \mathbb{Z}^+$. So, it is only the positive integers from the set of integers which will have a pre-image. The negative integers will have no pre-image. You cannot have $x^2 \in \mathbb{Z}^-$.

So, now the question is how do we prove whether a given arbitrary function is an onto function or not? So, we have to check whether this universal quantification is true for that given function. But if the domain and the co-domain of the function f is an infinite set, we cannot check whether this universal quantification is true for every b, and with respect to that b for some a.

So, what we will do is, instead we will follow the principle of universal generalization. Since we have to prove that the condition is true for all the elements b from the co-domain set, we will instead pick an arbitrary element b from the co-domain set. And for that arbitrarily chosen element b, we will show we will prove the existence of at least one pre-image element a from the domain such that $f(a) = b$.

And since we are going to we will show this for an arbitrary b from the co-domain set that will show that the condition holds for any element b from the co-domain set. That is all we will prove whether a given function is an onto function or not. Whereas if you want to prove that a function given function is not a surjective function, then we have to find a counterexample. Namely a counterexample for which the universal quantification does not hold. Then we have to find out element from the co-domain set say b which do not have any pre-image.

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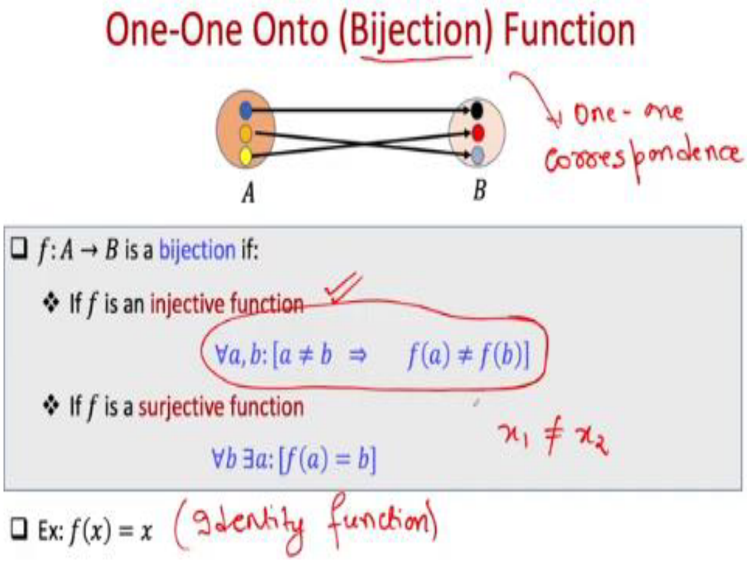

Now the third category of important functions is the one – one, onto function. They are also called as bijective functions. Sometimes they have also used a term one to one correspondence. There are many terms for the same concept. So, again you are given a function $f: A \rightarrow B$. It will be called as a bijective function or bijection provided f is an injective function. That means this condition should hold.

That means different element or distinct element should have different images and the function has to be a surjective function. So, if I take the function $f(x) = x$. This is also called as the identity function. Why it is called an identity function? Because every function, every element is just mapped to itself and it does not matter what is the domain and co-domain of this function. This will be always a bijective function.

Clearly it is an injective function because if you have $x_1 \neq x_2$, then $f(x_1) = x_1$, and $f(x_2) = x_2$. And since $x_1 \neq x_2, f(x_1) \neq f(x_2)$ that shows that the function is injective whereas if you take any given random or arbitrarily chosen element y from the co-domain set, the corresponding pre-image for that y is the element y itself. It shows that the function is a surjective function.

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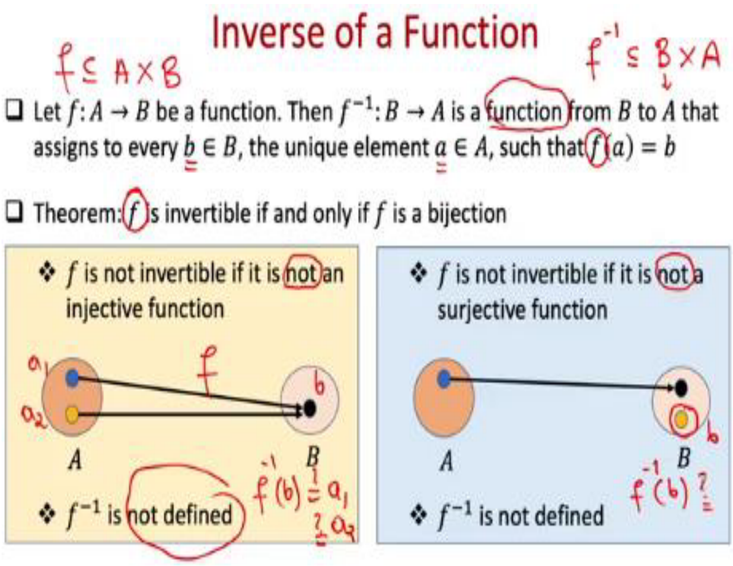

Now we will define what we call as the inverse of a function. So, imagine a function $f: A \rightarrow B$. So, since f is a special type of relation, remember it is a special type of relation we can find the inverse of that relation as well. So, basically we are trying to find out the inverse relation. If I denote as f^{-1} but it will be a function it will be defined as a function, that means it should satisfy the requirements of a function.

That means f inverse will be a subset of B cross A and it will be a function. That means each element $b \in B$ will appear as part of an ordered pair in this collection of ordered pairs of f^{-1}. And what will be the mapping of any element b as part of this inverse function? The mapping will be or the image of that element b will be an element a provided b was the image for that element the a as per the function f.

So, it is easy to see that a function f will be invertible if and only if the function is a bijection. If the function f is not a bijection then, we cannot define the inverse of that function. So, let us prove this formally. We can show that if your function f is not an injective function then clearly f is not invertible. So, since the function f is not invertible that means you have two elements in the domain say, $a_1 \neq a_2$, but getting mapped to the same b.

As far as the function definition of function is concerned this is allowed. I am not saying my function f is an injective function here. I am assuming my f is not an injective function. So, f is not an injective function that means this is possible. But if this is the case, what can you say about the inverse of b? Will you say $f^{-1}(b) = a_1$ or will you say $f^{-1}(b) = a_2$? it is ambiguous here.

We cannot find out the image of the element b as per the f^{-1} function. So, f^{-1} function will not be defined. I stress that if this would have been a relation then it is absolutely fine. It would have been a relation then in f^{-1} you could have (b, a_1) and you could have (b, a_2) as well. Because in relation there is no restriction, but f^{-1} has to be a function, and you cannot have two images for the same element if it is a function.

So, clearly this shows that if your function f is not an injective function then it is not invertible. We can also prove here that if the function f is not a surjective function then also it is not invertible. So, if it is not a surjective function that means you have some element at least one b from the co-domain set which do not have any pre-mage. If that is the case, then what can you say about the image of the element b as per the f^{-1} function? It is not defined here. But that is a violation of the definition of inverse of a function. So, that shows that a function is invertible if and only if f is a bijection.

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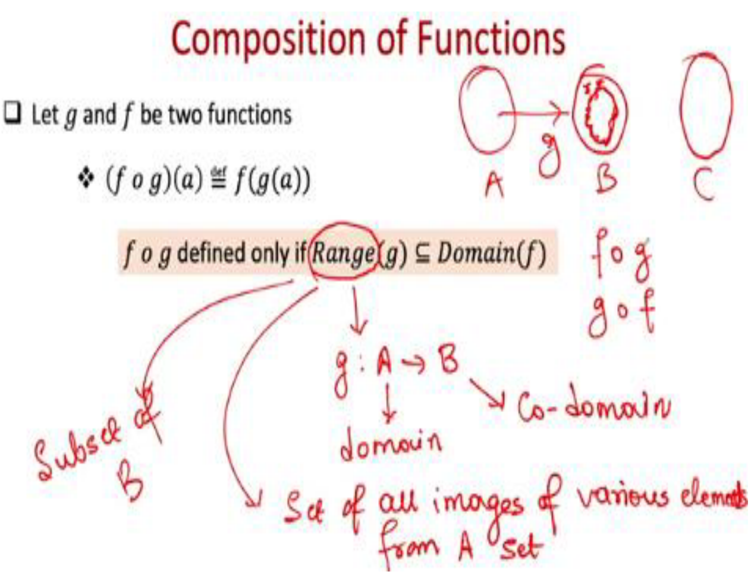

So, the last thing that we want to now define is the composition of functions. So, remember functions are special relations. So, since we can compose relations, we can compose functions as well provided certain conditions are satisfied. So, imagine you are given two functions, a function g and the function f with appropriate domain and co-domain. Then the composition of the f function and g function is denoted by fog.

So, let me make it more clear here. So, you imagine that $g: A \rightarrow B$ where $A = B$ or $A \neq B$. And $f: B \rightarrow C$. Then the composition of fog will be a function from the set $A \rightarrow C$. And what exactly will be that function. So, we will apply the function g first and obtain the images or possible elements a from the set A. And then we will apply the function f on those resultant elements and that will be the mapping of the element A as per the composition of f and g function, fog.

So, this composition of function has to be a function and it is easy to see that since it has to be a function then the composition of f and g is defined only if $range(g) \subseteq domain(f)$. So, what is exactly the $range(g)$ means?

So, since g is a mapping from I am assuming here that $g: A \rightarrow B$, then A is called as the domain and B is called as the co-domain. Then what is this range set? It is the set of all images of various elements from the A set. That means this range is a subset. It may be the entire set B or it might be a proper subset of the set B. So, if your function $g: A \rightarrow B$, what I am saying is I am just focusing on all the elements which is a subset of this set B which have pre-images.

That means they are the images of some element from the set A. Because your function g may not be a surjective function there might be some elements in the set B who do not have any pre- images. So, I ignore those elements when I am considering the range set. Those elements will be considered this s star elements will be considered as part of the co-domain set. But they would not be considered as part of the range set.

And when I say the range set, I am just focusing on the image set, the images that the function g assigned to various elements here. So, coming back to the composition of function, it is easy to see that fog, only if you take the range set of the g function then it should be a subset of the domain of the f function. If your $f: B \rightarrow C$, then if I focus on this circled thing, each of them should have an image as per the f function.

That means that circled thing should be a part of the domain set of the f function. If this condition is not satisfied and clearly the composition of fog is not defined. And again, the composition of functions need not be commutative in the sense $fog \neq gof$. First of all the composition of gof need not be defined at all, if fog is defined because A, B, C sets could be arbitrary. So, only under special conditions the compositions $fog = gof$. So, that brings me to the end of this lecture.

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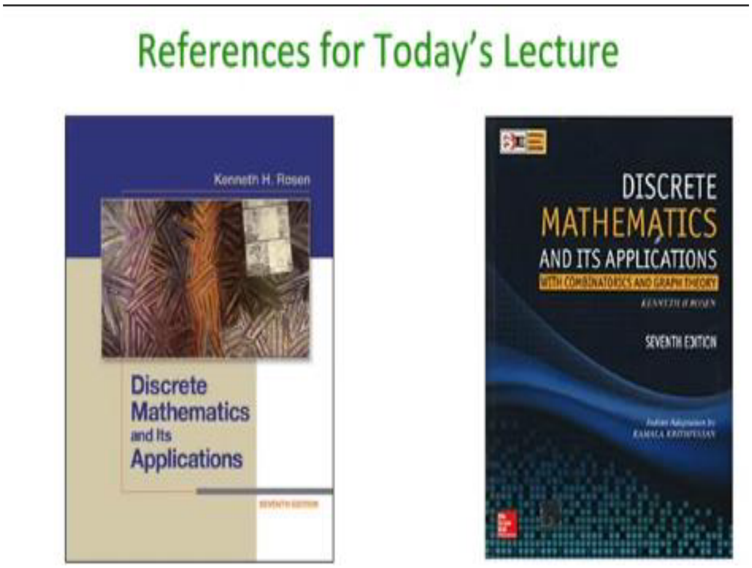

These are the references for today's lecture. Just to recap in this lecture we introduce the concept of set. We discussed about various types of various special types of sets and we also discussed about compositions of relations. Thank you!