Discrete Mathematics
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Lecture -02
Logical Equivalence

Hello everyone. Welcome to this lecture on logical equivalence. So, just a quick recap. In the last lecture we discussed about propositional logic, various logical operators.

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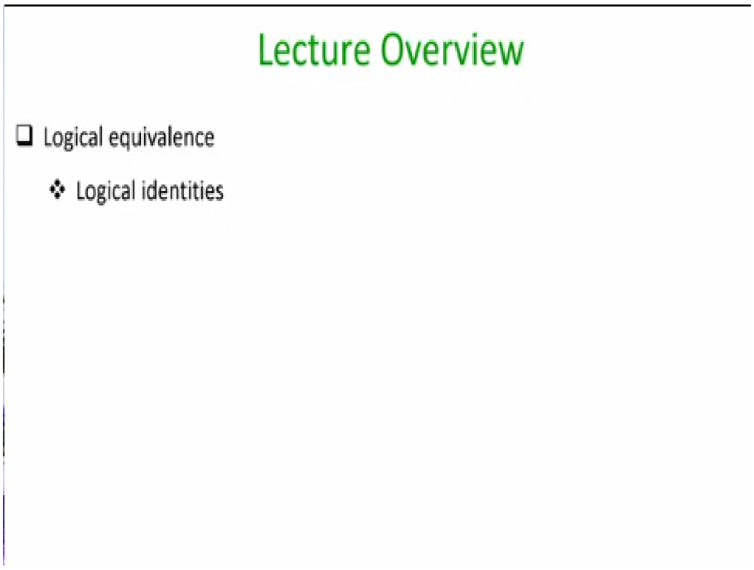

And how do we form compound propositions from simple propositions using logical operators. In this lecture, we will discuss about logical equivalence and logical identities.

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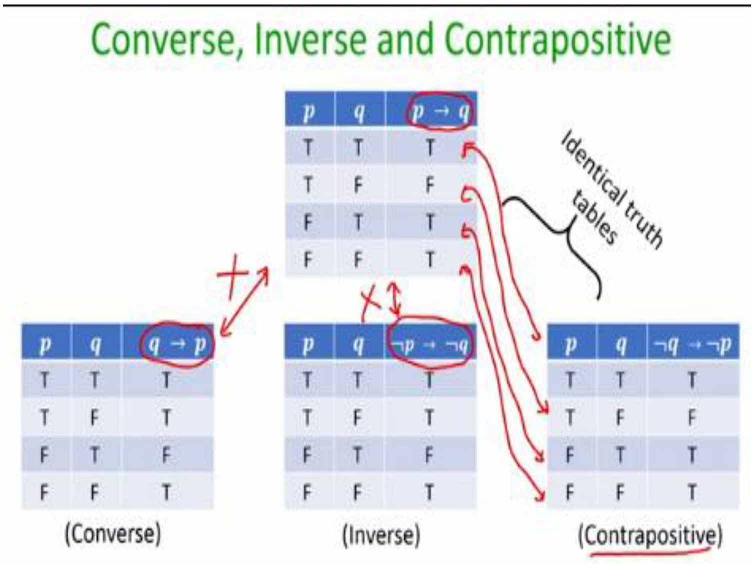

So, remember if p then q is represented by $p \rightarrow q$ and truth table of $p \rightarrow q$ is this. Then the converse of $p \rightarrow q$ is denoted by $q \rightarrow p$ and it is easy to see that the truth table of $q \rightarrow p$ or the converse is this. The inverse of $p \rightarrow q$ is denoted by $\neg p \rightarrow \neg q$ and its truth table will be like this and the contrapositive which is very important for $p \rightarrow q$ will be the statement $\neg q \rightarrow \neg p$.

And if you see closely, the truth tables of the converse of $p \rightarrow q$ and the truth table of $p \rightarrow q$, they are not same. If you see the truth table of $p \rightarrow q$ and inverse of $p \rightarrow q$ are also not same. But if you see the truth table of $p \rightarrow q$ and its contrapositive they are same; that means the first row of both the tables are same. The second rows of both the tables are same. The third rows of both the tables are same and same and same as fourth row.

And that is why I can say that $p \rightarrow q$ and negation $q \rightarrow$ negation p are the same statements, they are logically equivalent.

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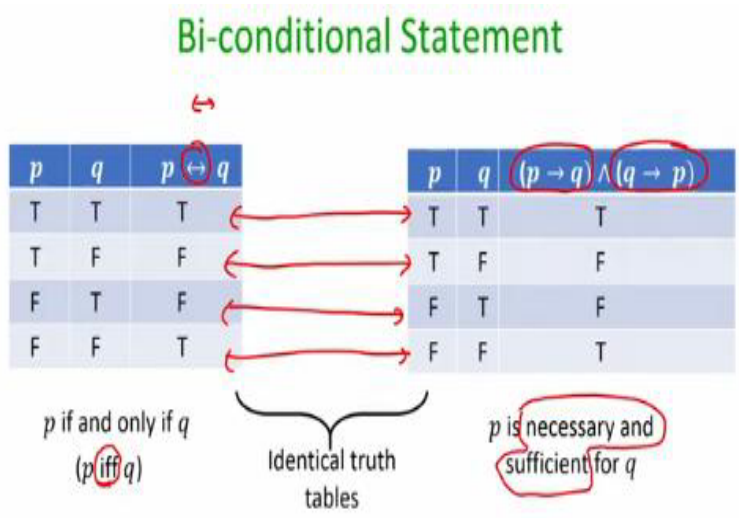

We will come back to that point later but let me first define a bi conditional operator or a bi conditional statement which for which we use this notation ↔ that means an arrowhead which has an arrowhead at both ends. And this bi conditional statement is used to represent statements of the form p if and only if q or in short form p if and only if q says another way another form of representing if and only if is iff.

So very often for mathematical and for various theorem statements, you must have seen conditions like prove that this is true if and only if this holds right? So wherever we are making statements of that form, we are actually making statements of the form p bi-implication q. Another equivalent form of this bi conditional statement is the conjunction of p implies q and q implies p.

So you can see that row wise, the first row of both the tables are same, the second row of both the tables are same. The third row of both the tables are same and the fourth row of both the tables are same. Hence I can say that this bi conditional statement is same as the conjunction of p →q and q →p. Now p →q means p is sufficient for q right? And q →p means p is necessary for q. So that is why this bi conditional statement also represents a statement of the form that p is necessary and sufficient for q.

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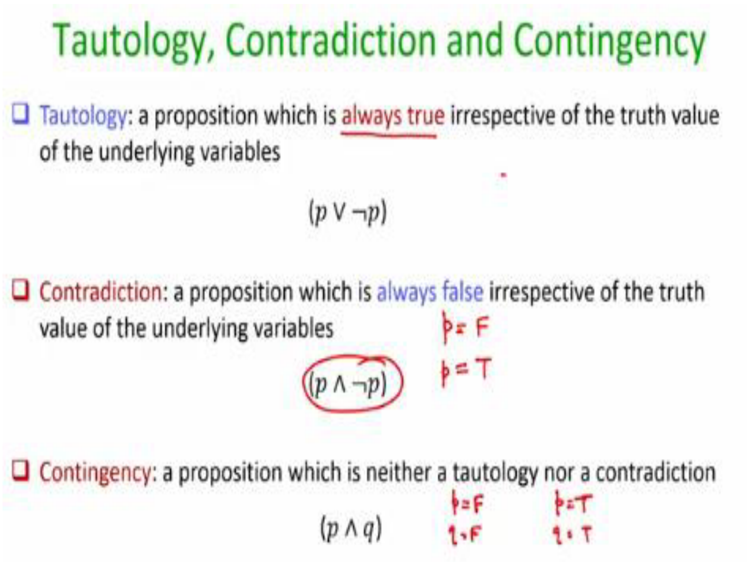

Now let us next define tautology, contradiction and contingency. So a tautology is a proposition which is always true, irrespective of what truth value you assigned to the underlying variables. So, for example, if I consider this compound statement namely the disjunction of p and ¬ p, then this will be always true; that means if p is true, then this is true and even for p equal to false this statement is again true.

That means it does not matter whether your p is true or false; this statement is disjunction of p and ¬ p will always be true and hence this is a tautology. Whereas a proposition is called a contradiction if it is always false irrespective of what truth value I assign to the underlying variables. So an example of contradiction is p conjunction ¬p. So you can verify that if p is false then this statement is false.

And even for p equal to true this overall statement is false that means this statement is always false for every possible truth assignment of p and hence it is a contradiction. Whereas a contingency is a proposition, which is neither a tautology nor a contradiction that means it can be sometime true it can be sometimes false. I cannot say that it is always true or it is always false. So for instance, if I take the statement p conjunction q then for p equal to false and q equal to false this overall statement is false. But for p equal to true and q equal to true, the statement is true. So, that is why it is a contingency.

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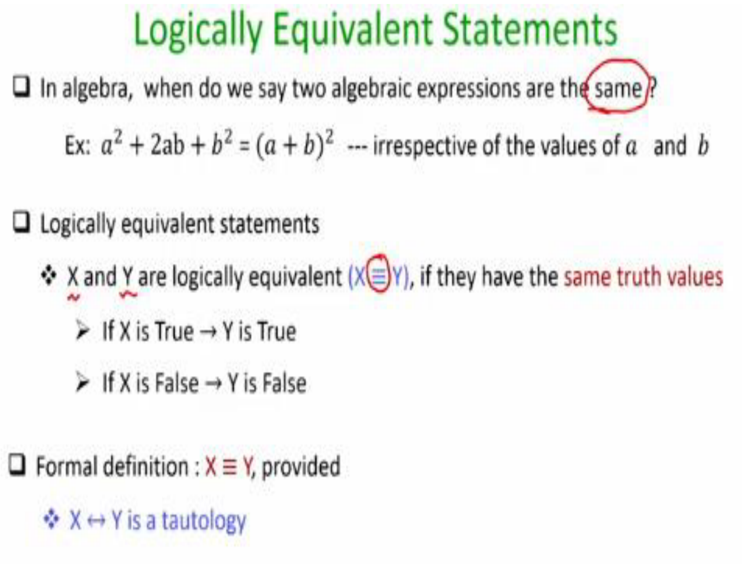

Now, we want to define what we call as logically equivalent statement. So before trying to understand what are logical equivalent statements? Remember in algebra and in mathematics, you often come across expressions of this form. We say for instance that $a^2 + 2ab + b^2$ is equal to $(a + b)^2$. That means these two expressions are the same expression. What do I mean by same expression? Well, by that I mean that whatever value you assign to a and b, the left hand side and right hand side will give you the same answer.

That is why these two expressions are the same expression. In the same way in mathematical logic if we have a compound proposition X and a compound proposition Y then I say that they are logically equivalent and I use this notation ≡. This is not an “equal to” notation, this is representation of equivalence, this is also called as an equivalence notation. So I say that X and Y are logically equivalent if they have the same truth values. What I mean by that is I mean that if X is true then Y is true if X is false then Y is false that means it never happens that X and Y takes different truth values.

More formally X is logically equivalent to Y provided the X bi-implication Y is a tautology, right? Because if X bi-implication Y is a tautology, then it means that whenever X is false Y has to be false whenever X is true Y has to be true. It cannot be possible that X and Y takes different values because if X and Y takes different values then the bi-implication of X and Y will be false and a tautology means that this statement is always true.

So the statement will be true only when both the sides of this expression or the compound propositions on both the sides take the same truth value.

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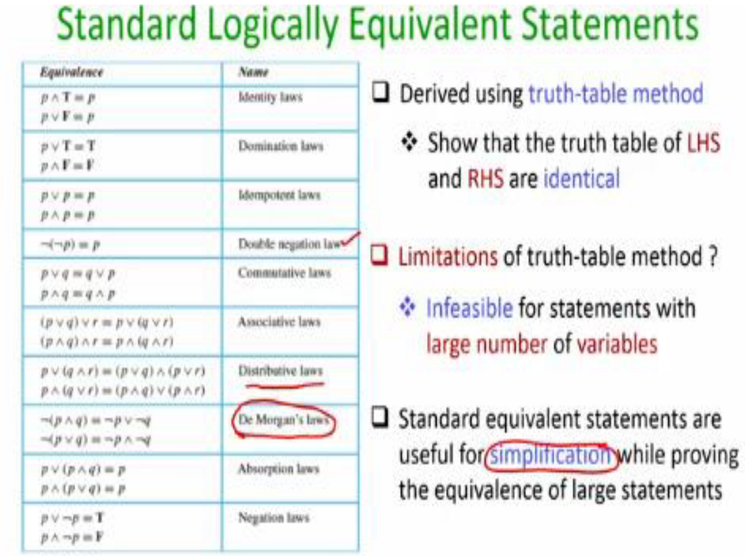

So there are various standard logical equivalent statements which are available which are very commonly used in mathematical logic and they are also called by various names. So for instance, the conjunction of p and true is always p that is called this law is called as the identity law. In the same way we have this double negation law which says that if you take the negation of negation of p then that is logically equivalent to p.

We have this De Morgan's law which is very important which says that if you have a negation outside then you can take the negation inside and split it across the various variables and if you have conjunction inside then it becomes disjunction and vice versa. We also have this distributive law this says that you can distribute the disjunction over conjunction and so on. How do we verify whether these logical identities are correct?

Well, we can verify using the truth table method namely we can draw, we can construct a truth table of the left hand side of the expression, we draw the truth table of the right hand side of the expression and verify whether the truth tables are the same. So for instance, if you want to verify the De Morgan's law, so the first part of the De Morgan's law says that the negation of conjunction of p and q is logically equivalent to negation p disjunction negation q. So what you can do is you can draw the truth table for the left hand side here.

You can draw the truth table for the right hand side part here. And you can easily verify that the rows of both the tables are equivalent, they are same and that is why I can say that these two are logical equivalent statement and now I have given a name namely De Morgan's law to this logical identity. However, the truth table method of verifying logically equivalent statement has a limitation.

Namely, the limitation here is it works as long as the number of variables the number of propositional variables which are there in your identity or the statement this is small. So in all this logical identity that I have written down in this table, there are at most three propositional variables and if I try to draw the truth table of a statement having 3 variables, and there will be only 8 rows which are easy to manage.

But imagine I have a logical identity which has a 20 number of variables then the number of rows and that truth table will be 2^{20} and definitely you cannot draw such a huge table. So that is why it is infeasible to verify the logical equivalence of statements using the truth table method and that is why what we do here is we use some standard logical equivalent statements.

So for instance, these are some of the standard logical equivalent statements, which we use to simplify complex expressions and verify whether those complex expressions are logically equivalent or not and this is something similar to what we do in our regular maths. In regular maths if we have two expressions and if you want to simplify one expression and convert it to another expression then we have some well-known rules which we can always use to do some substitution in our process of simplifying the expressions.

So we are trying to do the same thing even in the mathematical logic. If you are given a very complex expression X, a compound proposition X, which you want to show to be logical equivalent to Y and you do not want to involve the truth table method, then our goal will be to simplify the expression X and keep on doing the simplification till we can convert it into the expression which has the same form as Y.

During this conversion process or the simplification process we can use this well-known logical identities by just quoting their names. We do not have to separately prove the De Morgan's law because it is a well-known identity we can simply say that okay, we are using the De Morgan's law and hence we are substituting this part with this part and so on.

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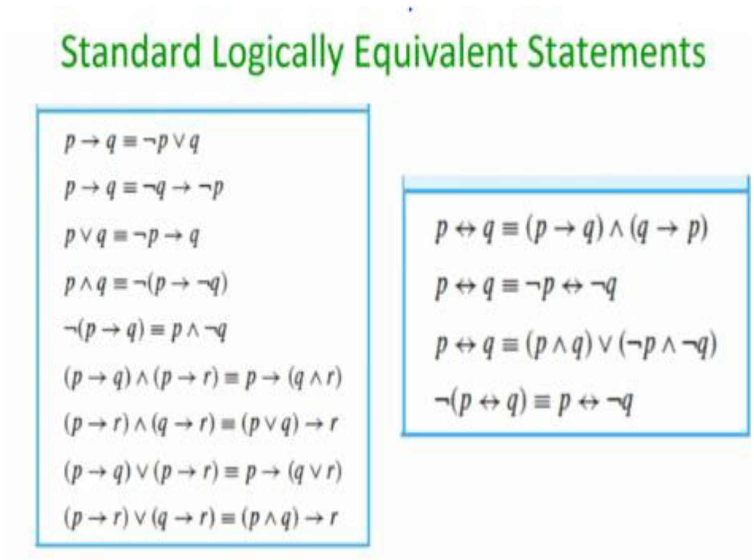

There are various other standard logically equivalent statements, so these are some of them they do not have any name but they are some well-known logical equivalent statements which we can use while doing the simplification.

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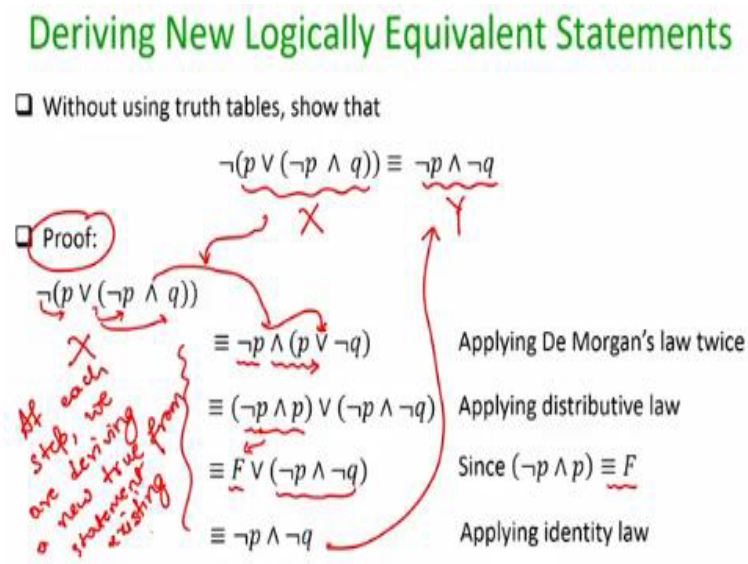

So now let us do an example here. Suppose I want to prove that my LHS expression and RHS expression, they are logically equivalent so this is my statement X this is my statement Y. Well in this case I can use the truth table method because my expressions X and Y involve only 2 variables and I can draw truth table which will have only 4 rows, but what I want to do here, I want to demonstrate here is that without even drawing the truth table, I can show that the expression X is logically equivalent to expression Y by using logical identities.

So here is the proof that expression X is equivalent to expression Y. I start with my expression X. What I can say is that this expression X is equivalent to this new expression and why this expression X is equivalent to this new expression because I can apply the De Morgan's law twice. So what I can do is I can take this negation first inside, so that is why I get this negation p and this negation is will be now present outside this bracket.

So that is why again I can apply the De Morgan's law and this negation when it goes inside the negation negation p becomes p and then this negation also goes to q. And this conjunction gets gets converted into disjunction. Now what I can say is that this expression which I have derived from the expression X can be further converted into this expression because I can apply the distributive law.

The distributive law says that you can always split the conjunction over disjunction, so that is what I am doing. Now I can say that this expression negation p conjunction p is equivalent to the value false. So we have this identity and I am not quoting the name of the identity but this is the well-known identity. So I can substitute this conjunction of negation p and p is false and whatever there is left over here and then I can apply identity law which says that the disjunction of false with any proposition is the proposition itself.

Now you can see that I started with X and I kept on applying various laws and I keep on doing the simplification and then finally I can derive expression Y and hence I come to the conclusion that starting with X, I can conclude a statement Y. And hence the statements X and Y are logically equivalent. So that is how we can derive new logically equivalent statement from old statements by applying well known logical identities and why this is called a proof, because at each step we are doing the following.

At each step we are deriving a new statement, a new true statement from the collection of existing true statement and this sequence of steps which I have done here constitutes what is called as a proof that indeed X is equivalent to Y. So that brings me to the end of this lecture. Just to summarize, in this lecture we introduced new logical operators namely the bi conditional operator, we introduced the terms tautology, contradiction contingency.

We defined what we call as logical equivalence of two statements. Two compound propositions are called logically equivalent to each other if they say take the same truth values or formally bi-implication of X and Y is a tautology. We discussed various well known logical identities which we can very quickly prove using truth table method and then we saw that how this well-known logical identities can be used to prove the equivalence of complex compound propositions by the simplification method where our goal will be to keep on simplifying the expression X and convert it into expression Y, thank you.