Discrete Mathematics
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Lecture -06
Tutorial 1: Part I

Hello everyone welcome to the first part of tutorial one, so let us start with question number 1.

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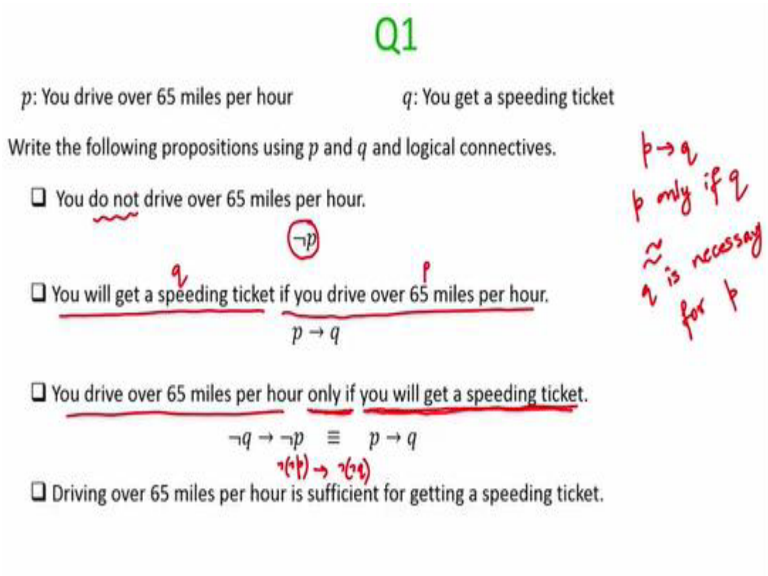

So in this question the goal is the following. You are given two propositional variables p and q representing the propositions you drive over 65 miles per hour and you get a speeding ticket respectively. Then your goal is the following using this two propositions p and q you have to represent the following statements and by compound propositions using logical connectives. So the first statement that we want to represent here is you do not drive over 65 miles per hour, which is very simple this statement is nothing but negation of p because p represents the statement you drive over 65 miles per hour, so you want to represent the negation of that. The second statement is that you want to represent here is you will get a speeding ticket if you drive over 65 miles per hour, so this is some form of if-then statement. This is the if part this is the conclusion. So the if part here is if you drive over 65 miles per hour which is p and the conclusion here is you will get a speeding ticket.

That is why this statement will be represented by $p \rightarrow q$. The third statement that we want to represent here is you drive over 65 miles per hour only if you will get a speeding ticket. So this is a statement of the form only if, so recall $p \rightarrow q$ also represents p only if q, or equivalently q is necessary for p. These are the various forms for $p \rightarrow q$. So this condition is the necessity condition here.

So you can write it, you can write the statement either in the form of $\neg q \rightarrow \neg p$. Why $\neg q \rightarrow \neg p$? Because remember when I say only if part that means whatever is there after only if part if that condition is not satisfied, What is that condition, you will get a speeding ticket, which is your statement q. So the equivalent form of only if is that if that negation of that thing happens, then whatever is there before only if that does not happen.

And what is there before only if is the statement that you drive over 65 miles per hour, which is your p. So that will not happen. So that is why this statement can be represented by $\neg q \rightarrow \neg p$ and remember that the contra positive of an implication is equivalent to the original implication. So what will be the contra positive of this implication?

So it will be negation of negation of p implies the negation of negation of q and if you take negation inside you get $p \rightarrow q$. The last statement we want to represent here is driving over 65 miles per hour is sufficient for getting a speeding ticket. That means whatever is there before your sufficient part that is your if statement. If you ensure that then whatever is thereafter sufficient that will be ensured that will happen. So this is equivalent to $p \rightarrow q$. So it is a very simple straightforward question.

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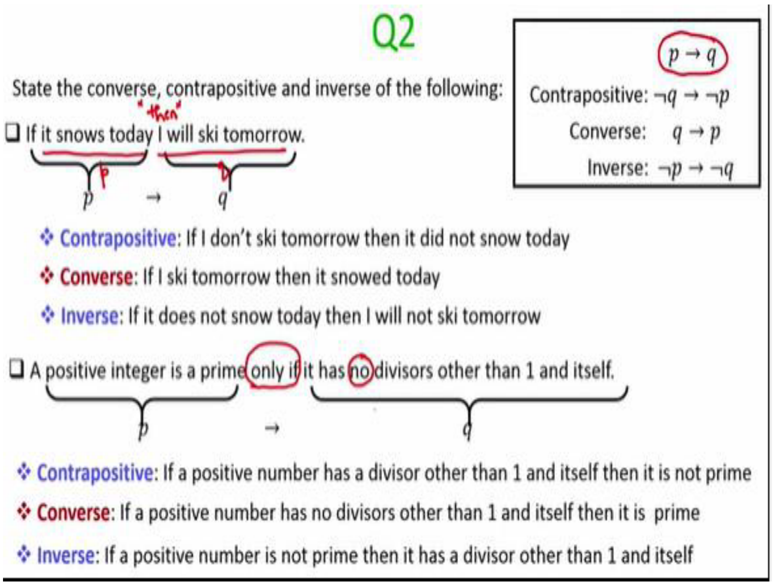

In question 2 the goal is the following, you have to write down the converse contrapositive and inverse of the following statements. So just to recall, if you are given an implication $p \to q$ then the contrapositive of that is $\neg q \to \neg p$, the converse of that implication is $q \to p$ and inverse of that implication is $\neg p \to \neg q$, that is the definition.

Now the first statement here is if it snows today I will ski tomorrow. So this is your p part and this is your q part. The statement then is not explicit sorry the word then is not explicitly given here but it is present implicitly here. So this is your $p \to q$ part. So the contrapositive of this will be $\neg q \to \neg p$ and $\neg q$ will be if I do not ski tomorrow, $\neg p$ will be it did not snow today.

The converse will be $q \to p$; that means if I ski tomorrow then it snows today, the inverse will be $\neg p \to \neg q$. So $\neg p$ means it does not snow today $\neg q$ means I will not ski tomorrow. The second statement here is a positive integer is a prime only if it has no divisors other than 1 and itself. So let us first identify the implication here. What is the form, what is the implication here in what?

So this is an only if statement. So you can represent whatever is there before only if as p whatever is there after only if as q and the implication that is here is $p \to q$. The contrapositive will be $\neg q \to \neg p$. So the negation of q will be the number has a divisor other than one and itself because q means it has no divisors. So negation of q means, it will have a divisor other than one and itself. And p is the number is a prime so negation of p will be it is not a prime, so straight forward.

Similarly the converse will be q → p so q is the number has no divisors other than one and itself, p is the number is a prime. The inverse will be ¬ p → ¬q so ¬ p means the number is not prime. If the number is not prime then we have to conclude ¬ q, ¬ q means it will have a divisor other than one and itself. So that is how you write the contrapositive of converse and inverse of implications.

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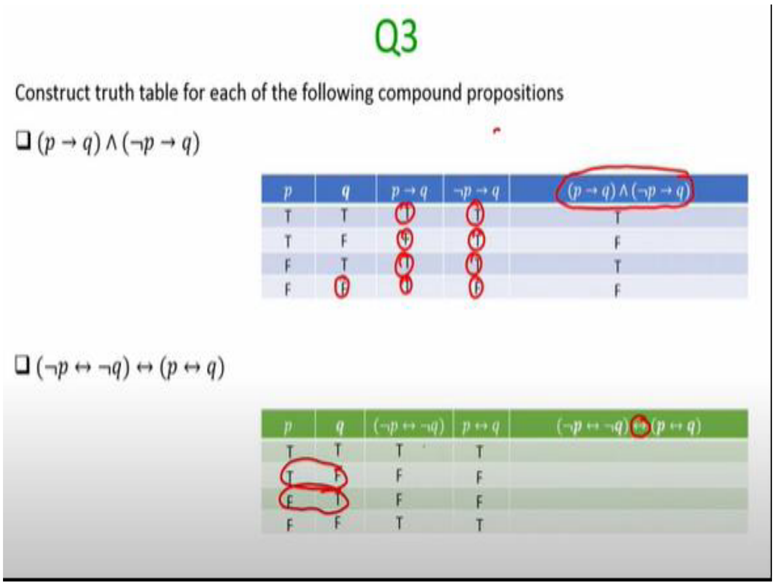

The question 3 is asking you to do the following. You are given, a set of compound propositions and you have to draw the truth table it is a very straightforward question here. So the first compound proposition is this conjunction of two implications. What you have to do is it is a compound proposition involving two variables p and q so you have one column for all possible values for p one column for all possible values of q and then what I am doing here is for simplification, I am separately writing down the column for p → q. I am separately writing down the column for ¬ p → q and then finally I am separately writing down the column for conjunction of these two things namely p → q and ¬ p → q. So let us first begin with the column for p and q. I have written down the various possible truth assignments that p and q can take.

Based on this, the column for p → q will be this so remember p → q takes the value false only when p is true and q is false. For all other possible assignments p → q is always true now the next column is for ¬ p → q and the column will take these values. So let us focus on the entry when this statement is false. So this statement will be false when LHS is true, but RHS is false.

That means your q is false and p is false. Because if p is false then this negation of p will become true, and that is why this true implies false will lead to the value false for all other three combinations, this implication will always take the truth value as true. Now I have the columns for $p \rightarrow q$ and negation $p \rightarrow q$ what I have to do is I have to take the conjunction.

And remember conjunction of two things two variables will be false if any of them is false. So the truth value for the conjunction will be as follows, since I have true here and true here and true is true, but I have false here and true here so conjunction will be false. I have true here and true here, they are conjunction will be true. I have true here and false here and the conjunction will be false.

That is how you can build a truth table for this compound proposition. The second compound proposition for which I am supposed to draw the truth table is this, it involves several bi-implications. So I can do the similar stuff which I have done for the first part. I will be separately writing down the column for p. I will be separately writing down the column for q and I will be separately writing down the columns for each individual portions of this compound proposition, and this will be the final column, which I am interested in.

So the columns for p and q are straight forward based on the truth values that I am giving to p and q the column for this bi-implication, the first bi-implication is this. So remember bi-implication is going to take the truth value true if both the LHS and RHS have the same truth value. If both of them are false or if both of them are true the bi- implication will take the value true.

So that is why wherever you have a mismatch, the bi-implication takes the value false; otherwise the bi implication takes the truth value true. Similarly you can draw the column for the $p \leftrightarrow q$, wherever there is a mismatch you get the false value otherwise it is true. And then you have to take the bi-implication of this column and this column. Wherever they are matching, they will take the truth value. In fact, they are matching at all positions all the four positions and that is why it will be always true.

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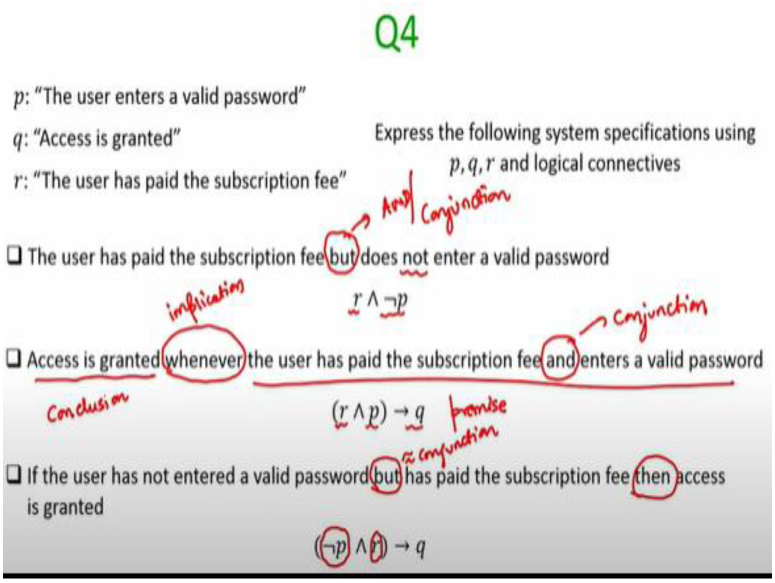

Now, the fourth question is the following. You are given three propositional variables p, q, r denoting these three statements. And using p, q, r and various logical operators, connectives, you have to write down compound propositions representing some given English statements. So the first statement is the user has paid the subscription fee but does not enter a valid password.

So you might be wondering what will be the logical connective operator for representing 'but'. So the but here should be treated as some form of and or conjunction here. That means, you can equivalently pass the statement saying that user has paid the subscription fee and he has not entered a valid password. And I know that the variable r represents the statement user has paid the subscription fee.

The variable p represent the user enters a valid password, but I want to represent here, denote here, state here that he has not entered a valid password, that is why it will be ¬ p. The second statement is access is granted whenever the user has paid the subscription fee and enters a valid password. So let us parse first this statement. So remember 'whenever'- is a form of Implication.

That means this part is going to be a premise and this is the conclusion. So that means if this is a form of if-then, so whatever is there after whenever that is the if-part, if that thing is ensured then whatever is there before whenever that is the conclusion, that you can conclude. Now in the premise here a conjunction is involved because my premise consists of conjunction of two things.

And now you have all the details to write this statement in the form of a compound proposition, so my premise here is the conjunction of two things, the user has paid the subscription fee which is denoted by variable r and he has entered a valid password which is p. That is the conjunction of these two things is the premise if both these two things are ensured the conclusion that I can draw is that access will be granted and access is granted means q.

The last statement is the user has not entered a valid password but has paid the subscription fee then access is granted. So this is a form of if-then statement. Whatever is there after then that is the conclusion whatever is there before then that is a premise, but in the premise you again have an occurrence of 'but' and remember but is nothing but conjunction. So you can represent this statement in this form.

The premise is the conjunction of two things namely user has not entered a valid password, so user has entered a valid password is p, so he has not entered a valid password is ¬ p. But he has paid the subscription fee that means, the statement that he has paid the subscription fees will be represented by r and as I said 'but' should be treated as a form of conjunction so premise is ¬ p conjunction r.

The conclusion that I want to draw here is that in that case access is granted. Access is granted is denoted by q.

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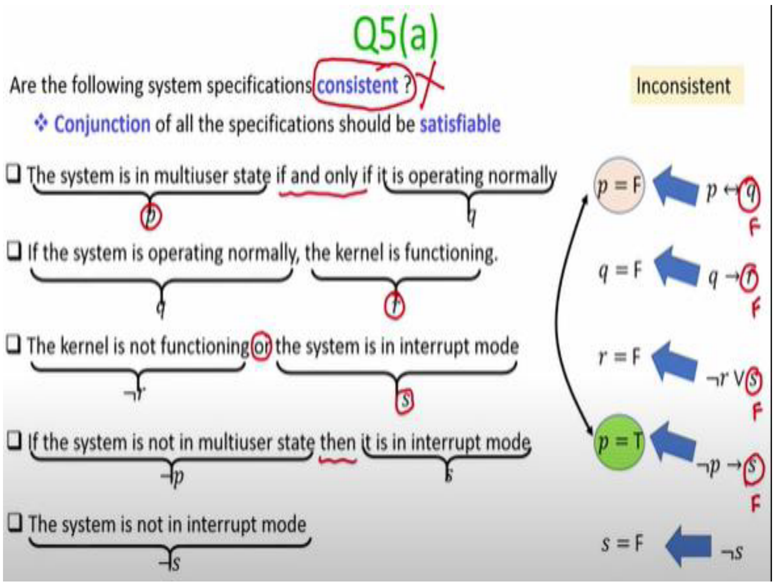

So now let us consider the first part of question 5. I would not be discussing the second part of the question 5. I will be just discussing the first part the second part I am leaving for you as an assignment. So the question basically says the following that you are given a set of statements and you have to verify whether the set of statements constitute a consistent system specification and what exactly I mean by a consistent system specification?

Well by that I mean that is it possible to simultaneously satisfy all those conditions that means is it possible that each of the individual conditions simultaneously hold. Or logically the conjunction of all this specification should be satisfiable. Because if the conjunction of all those specifications is satisfiable that means there is some truth value which you can assign which can satisfy each of those individual specification.

So what are the specifications that are given to you? You are given here 5 statements, 5 specifications. The first thing we have to do here is we have to convert this thing into an abstract argument form or sorry in the form of compound propositions. So for that we will introduce variables to represent the various statements here. So let p represent the statement the system is in multi user state and let q represent a statement it is operating normally.

So the first statement is a form of if and only if, that means bi-implication. So the first statement can be represented as $p \leftrightarrow q$ then the second statement or the specification requires two new variables q and r. So let q represent the statement proposition the system is operating normally and let r represent the proposition the kernel is functioning. Then the second statement is a form of if-then statement which can be represented by $q \rightarrow r$.

Then the proposition kernel is not functioning can be represented by $\neg r$ because I have already used the variable r to represent a statement the kernel is functioning. So the kernel is not functioning will be represented by negation of r but I have to introduce a new variable to represent proposition the system is in interrupt state because I have not encountered it earlier.

And this is a disjunction because it is an or statement. So this statement can be logically represented as distinction of $\neg r$ and s. I do not need any new variable for the fourth statement because I have already used a variable p to represent the proposition system is in multi user state, so system is not in the multi user state will be represented by $\neg p$.

And I have already used variable s to define, represent a proposition the system is in an interrupt mode. So the system is in interrupt mode will be represented by s and this is an if-then statement so it will be $\neg p \rightarrow s$. And the last statement is the system is not in the interrupt mode. So system is an interrupt mode is represented by s. So this statement will be represented by negation of s.

So now I have to verify whether all this 5 compound propositions can be satisfied simultaneously. So if at all negation of s has to be true, that means the variable s should be false. Now if variable s is false and I want this implication $\neg p \rightarrow s$ to be true that is possible only if p is true otherwise I will not be able to satisfy the statement $\neg p \rightarrow s$.

Now since s is false in order to satisfy this disjunction namely the disjunction of $\neg r$ and s, my r has to be false because if r is true and s is also false, then this disjunction can never be satisfied, so my r has to be false. Now my r is false then in order to satisfy this proposition $q \rightarrow r$, my q has to be false because if q is true and r is false then this proposition $q \rightarrow r$ can never be satisfied.

And now if my q is false, then this bi-implication $p \leftrightarrow q$ can be satisfied provided, p is false. But here is a contradiction. In order to satisfy $\neg p \rightarrow s$, my p should be true but in order to satisfy $p \leftrightarrow q$, my p should be false. But p cannot simultaneously take the value true as well as false that means I can conclude that there is no possible truth assignment for p, q, r and s, which can simultaneously satisfy or ensure that all the five statements here are true that means this system specification is not consistent. You cannot have a system where all this five conditions simultaneously, hold. The similar exercise you can do for the part b of question five. You are given a set of statements about a system namely a set of specifications and you have to convert those specifications into compound propositions and then you have to check whether the conjunction of those compound propositions is satisfiable or not.

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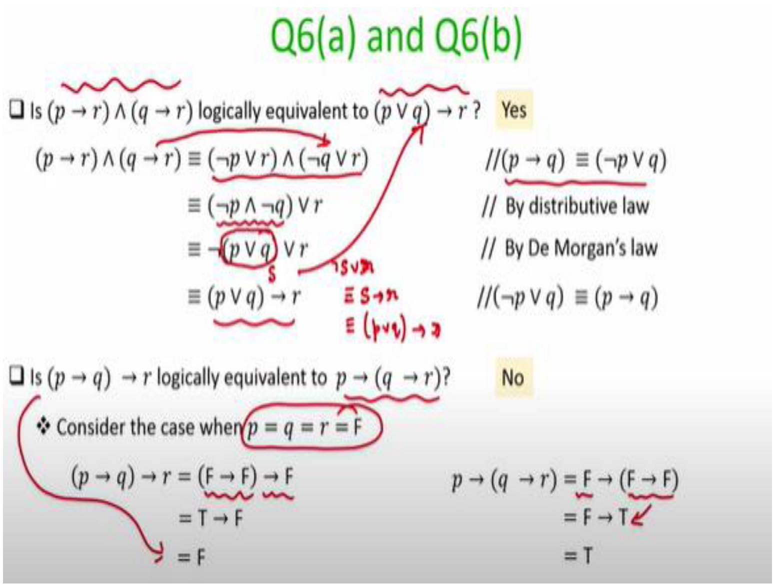

Ok now let us go to question 6, we will discuss part a and part b of question 6. Part a of the question is we have to verify whether the conjunction of p → r and q → r is logically equivalent to the implication p or q → r. So again you can use truth table method you can draw the truth table for that LHS part here, you can draw the truth table for this RHS part and then check whether both the truth tables are same or not.

We will not do that we want to apply various identities, rules of inferences and so on. So what we will do is we start with the LHS namely p → r conjunction q → r. Somehow I will try to bring it into my RHS part. So what I can do is I can replace this p → r by ¬ p or r because I know that p → q is equivalent to the disjunction of ¬ p and q.

And the same rule I can apply here for converting q → r into a disjunction of ¬ q and r. Now what I can do is. I can apply the distributive law and simplify the conjunction of these two clauses I can bring it into this form. Because if indeed I apply the distributive law the disjunction goes inside and r also goes once with ¬ p and once with ¬ q. Now what I can do is I can apply the De Morgan's law and write this conjunction of ¬ p and ¬ q in the form of negation of this whole disjunction.

And then again, I can apply this law namely, ¬ p or q is equivalent to p → q here. So you can imagine that, this whole thing is some s and r. So this is the form of ¬ s or r and this is equivalent to s → r and then you can substitute back s to be p or q. And what is this? This is nothing but your RHS that means starting with LHS by keep on simplifying it we can convert it into our RHS form and hence I can conclude that my LHS and RHS are logically equivalent.

The second part of this question is we have to verify whether $(p \rightarrow q) \rightarrow r$ is logically equivalent to $p \rightarrow (q \rightarrow r)$. So I have explicitly added the parenthesis here. Because the parenthesis says in what order the implication is going to be applied. It turns out that these two statements are not logically equivalent and we can prove it by a counter example namely we can give we can demonstrate a truth assignment when for that particular truth assignment the two statements take different truth values.

There could be many such counter examples, if at all you want to show that the two statements are not logically equivalent, even if you show one of the counter examples that is sufficient. So the counter example or the truth value that I will demonstrating here is when p q and r all takes the truth value false. In that case, the left hand side part namely $p \rightarrow q \rightarrow r$ will be considered as false implies false and then that implies false.

But false implies false is true and true implies false is false. So that means for the truth assignment that I have considered here, this implication is going to take the value false. Now, what about this implication; this implication is $p \rightarrow q \rightarrow r$. So p is false and $q \rightarrow r$ is this false implies false. The false implies false is true and false implies true is true.

That means what I have demonstrated here is that for the truth assignment when p, q and r are all false the two expressions have different truth assignments or truth values. One of the expressions takes the value false the other expression takes the value true and hence they cannot be logically equivalent because as per the definition of logical equivalence both of them should have the same truth value.

So I am leaving the other parts of question 6, you can verify similarly whether the two statements are logically equivalent or not?

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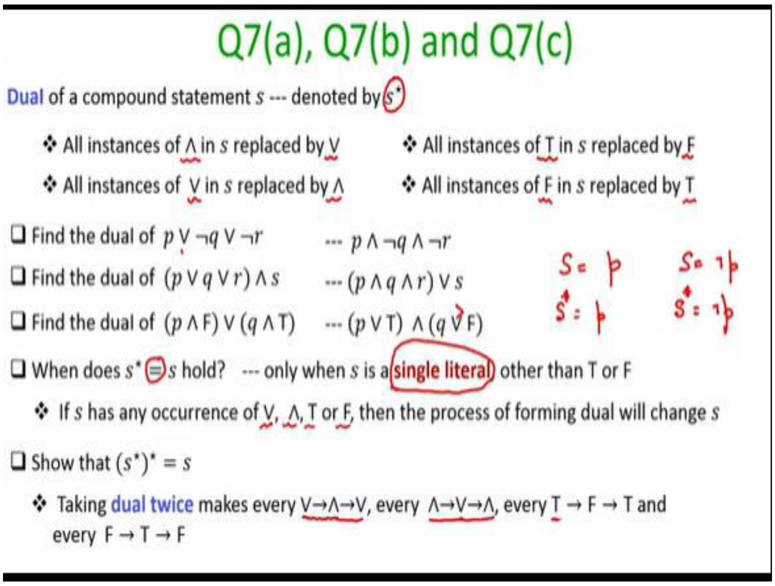

So let us go to question 7, in question 7 we are defining a concept which we call as the dual of a compound proposition. Dual of a compound proposition is denoted by this notation s* and what exactly is the dual? How do we construct a dual of a compound proposition? What we have to do is wherever we have an occurrence of conjunction in s we replace them by a disjunction.

Wherever there is a disjunction we replace them by a conjunction. Wherever there is an occurrence of the constant true, we replace them by constant false and wherever there is an occurrence of false I replace them by constant true. If you apply these 4 rules throughout the expression s then the resultant expression that you obtain is called as s*. So the first part of the question is you are given some statements and you have to construct their duals. So here is one of the compound propositions.

So what I have to do is remember while forming the dual I do not change the literals, the literal remains in their original form. I just have to change the conjunctions and disjunctions and the constants. So this distinction becomes conjunction and this disjunction becomes conjunction that is all. This will be the dual. In the same way the dual of the second statement will be this disjunction goes to conjunction, this disjunction goes to conjunction and this conjunction goes to disjunction.

The third statement you have now some constants also involved. So the conjunctions and disjunctions are converted vice versa and now you have false getting converted to true and true getting converted to false, that will be the dual. Now the b part of question 7 was you to ask the following it says when is it possible that the dual of the statement is exactly equal to the original statement?

I stress I am asking here exactly equal that means structurally, formula wise it is exactly the same proposition as the original proposition I am not saying logically equivalent and answer is very simple its, s* will be equal to s only when s is a single literal and that too different from the constants true or false, why so? Because if the compound proposition s has any occurrence of conjunction, disjunction, constant true, constant false then when you form the dual of that statement you will get a different expression.

Because all the conjunctions will be replaced by disjunctions, disjunctions by conjunction, true by false, false by true and so on. But if your statement s is, a statement just a single literal say p or say if s is equal to ¬ p then in that case s* will be same as your original s. For all other cases s* can never be equal to s. The third part of question 7 ask you to show that if you take the dual of a statement and then again take its dual you will get back the original expression and it is very simple.

Because if you take the dual twice then when you are taking the dual first time all the disjunctions get converted to conjunctions and then again when you take the dual second time conjunctions are converted back to disjunctions. The same thing happens for conjunctions when you take the dual first time they get converted to disjunctions and again when you take the dual second time they get converted to conjunction.

Similarly if you have any occurrence of constant true first time when you form a dual they get converted to constants false and again when you take the dual they get back to constant true and similarly for false this is very straightforward.

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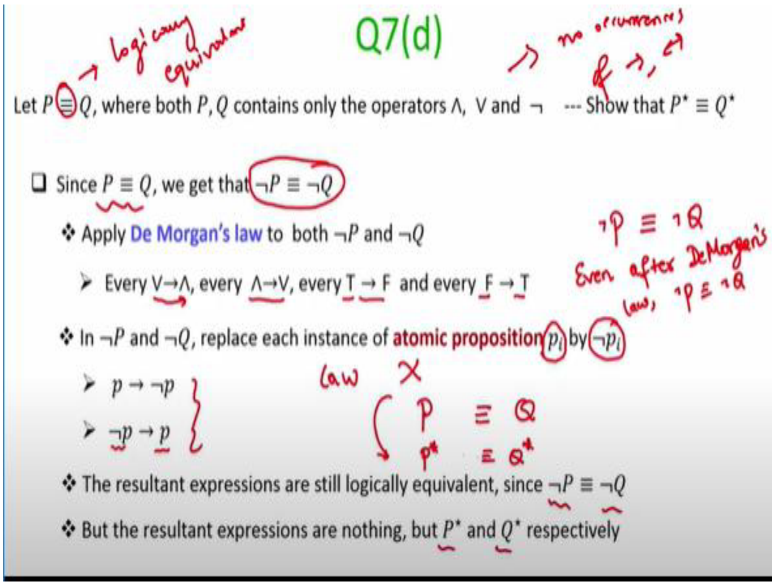

The last part of the question 7 is the following. You are given two compound propositions P and Q and they are logically equivalent it is given to you and it is also given that P and Q contains only conjunction, disjunction and negation there is no occurrence of implications and bi-implication. No occurrences of implication, bi-implication. In that case, we have to show that the dual of P and dual of Q are also logically equivalent.

So here is how we can prove that. So since P and Q are logically equivalent their negations also will be logically equivalent because if the negation of P and negation Q are not logically equivalent that means they take different truth values then how can it is possible that P and Q are logical equivalent. When I say P and Q are logically equivalent that means they take the same truth value.

Whenever P is true Q is true whenever P is false Q is false. That is a definition of logical equivalence. Now what I can do is let me apply the De Morgan's law both to the expression (¬ P) as well as to the expression (¬ Q). In this process what will happen is the following : all the occurrences of disjunction gets converted to conjunction all the occurrences of conjunction gets converted to disjunction all occurrences of constant true get converted into false and similarly all occurrences of false get converted into constant true.

This will be effect of applying the De Morgan's law and remember the ¬ P is logically equivalent to ¬ Q then even after applying the De Morgan's law to ¬ P and ¬ Q the resultant expressions will remain the same because I am not doing anything fancy here. I am just applying some standard identity even after applying De Morgan law, ¬ P will be equivalent to ¬ Q.

But the next thing that I do is, in ¬ P and in ¬ Q each occurrence of the atomic proposition p_i or a literal p_i is replaced by negation of that literal. That means wherever you have an occurrence of small p, in p and q you replace them by ¬ p and you do this both in negation of P as well as in negation of Q, right. In the same way wherever you have an occurrence of negation of propositional variable p in this expressions negation of p and negation of q you replace them by positive p.

Now, if you closely observe what we have done is by applying the De Morgan's law and by replacing small p by this negation of small p and by negation of small p by positive p we have obtained expressions which are P* and Q* and throughout this process the logical equivalence of my LHS part and RHS part is maintained. Because I started with two logical equivalent statement P and Q their negations will be logically equivalent.

Then individually if I apply the De Morgan's law in the ¬ P part and ¬ Q part the resultant expressions will be still logically equivalent and now if I apply this substitution of replacing each atomic proposition by its negative form simultaneously in the left hand side part and in the right hand side part the resultant expressions will be still logically equivalent. But in this whole process starting with P and Q, I have got down P to P* and Q to Q* and the logical equivalence of both the LHS part and RHS part is retained.

And that shows that if you start with two logically equivalent statement and if you take their duals, their duals also will be logically equivalent and this is a very powerful result because what it says is that if you have some well known identity established with respect to P and Q then the same law is applicable even for P* and Q* namely if you have proved a law say law X. I do not know what, law X could be anything.

This says that hey expression P is equivalent to expression Q then you can get another form of the law X where you say that P* is equivalent to Q*. You do not have to separately write down as law X.2 or law X.3 for the dual part. It comes automatically for free because of this result that the dual of two equivalent statements are also equivalent. So with that, I conclude the first part of tutorial 1. Thank you.