Discrete Mathematics
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Lecture -22
Equivalence Relations and Partitions

Hello everyone, welcome to this lecture on equivalence relations and partitions.

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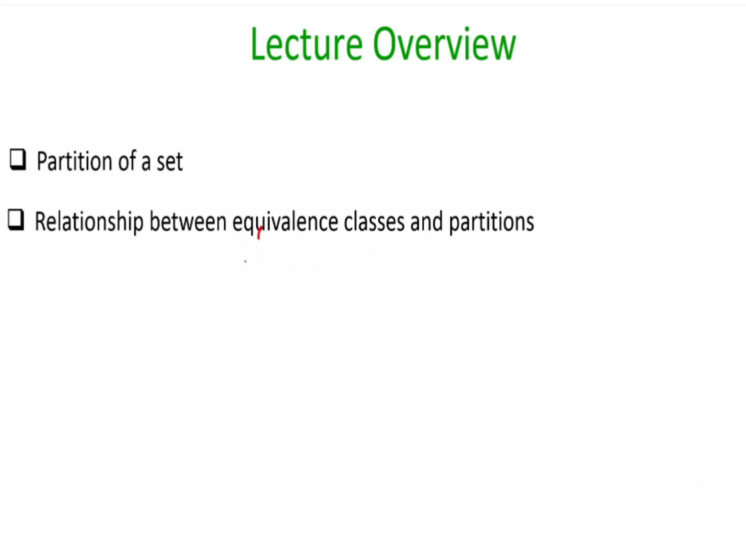

Just to recap in the last lecture we introduced the notion of equivalence relation and equivalence classes. In this lecture, we will continue the discussion on equivalence relations and classes. And we will introduce the notion of partition of a set and we will see the relationship between equivalence classes and partitions.

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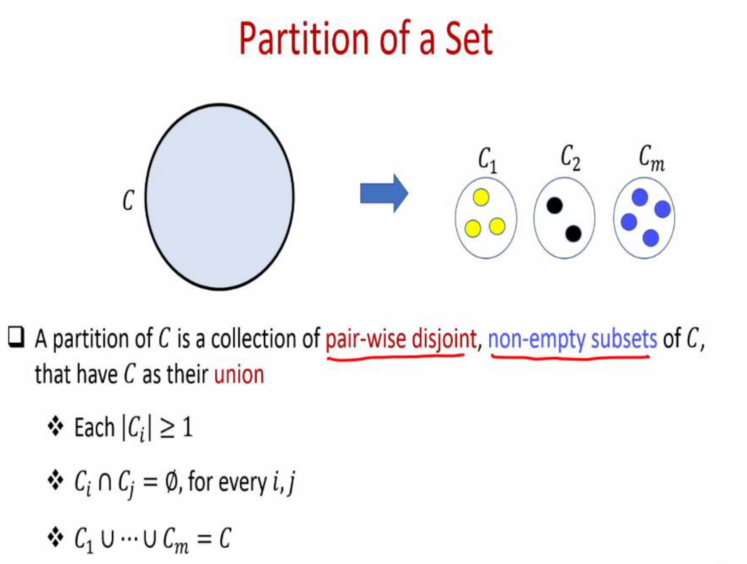

So, let us start with the definition of a partition of a set. So, imagine you are given a set C which may be finite or it may be infinite. Now, what is the partition of this set C? The partition here is basically a collection of pairwise disjoint, non-empty subsets say m subsets of C which should be pairwise disjoint such that if you take their union, you should get back the original set C.

So intuitively, say for example, you have the map of India you can say that the various states of India partition the entire country India into various subsets such that there is no intersection among the states here. So, in that sense, I am just trying to find out some subsets of the set C such that there should not be any overlap among those subsets and if I take the union of all those subsets I should get back the original set C, there should not be any element of C which is missing.

So, more formally the requirements here are the following. Each subset $C_i \neq \phi$ that means each subset should have at least one element. They should be pairwise disjoint. That means if I take any i, j then $C_i \cap C_j = \phi$ and $C_1 \cup \ldots \cup C_m = C$. So, one trivial partition of the set C is the set C itself.

I can imagine that C is partitioned into just one subset namely the entire set C or I can decide to partition C into exactly two halves or I can decide to partition C into three equal sets of equal sizes and so on. So, there might be various ways of partitioning your set is not a unique way of partitioning a set. Of course how many ways you can partition a set that is a very interesting question we will come back to that question later.

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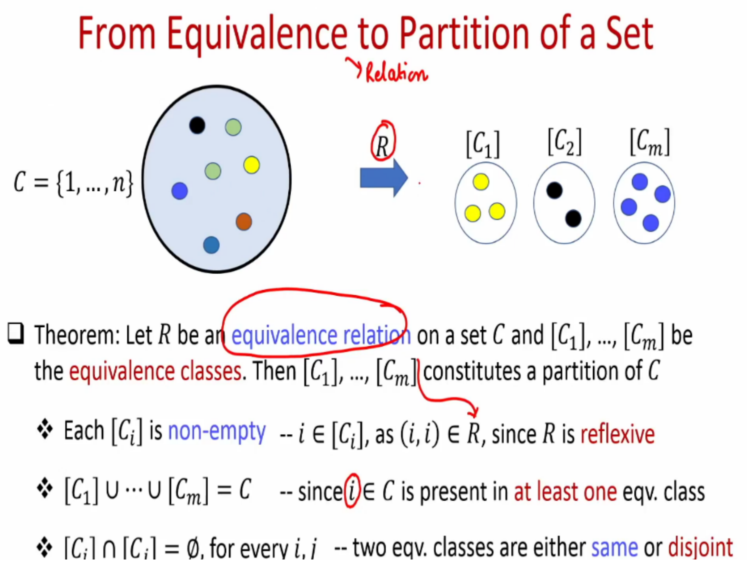

What we now want to establish here is a very interesting relationship between the equivalence from an equivalence relation to the partition of a set. So, we want to establish relationship between equivalence relation and partition of a set. So, imagine you are given a set C consisting of n elements. Now what I can prove here is that if R is an equivalence relation over the set C and if the equivalence classes which I can form with respect to the relation R are $C_1, \dots, C_m$. Then my claim here is that the equivalence classes $C_1, \dots, C_m$ constitutes partition of the set C.

So, just to recall, the definition of partition demands me to prove three properties, the first property is that each of this subset should be non-empty. And that is trivial because I know that each of these equivalence classes is non-empty because each of these equivalence classes is bound to have at least one element, $i \in [C_i]$ since my relation R is an equivalence relation, it will be a reflexive relation that means the element i will be related to itself. That means none of these equivalence classes will be an empty set. So, the first requirement is satisfied.

The second requirement from the partition is that the union of the various subsets should give me back the original set. So, my claim here is that if I take the union of all these m equivalence classes, I will definitely get back my original set C. And this is because you take any element $i \in C$, it is bound to be present in at least one equivalence class. Specifically, the element $i \in [C_i]$. So, that means I can safely say that if I take the union of these m equivalence classes, I will not be losing any element of the set C.

Third requirement from the partition was that the various subsets in the partition should be pairwise disjoint. So, in this specific case, I have to show that you take any two equivalence classes, they should be pairwise disjoint and that is easy because in the last lecture we proved that two equivalence classes are either same or they are disjoint. You cannot have a common element present in two different equivalence classes which automatically establishes that these subsets are pairwise disjoint. So, we have proved here that you give me any equivalence relation and if I take the equivalence classes that I can form with respect to that relation R that collection of equivalence classes will constitute a partition of my original set.

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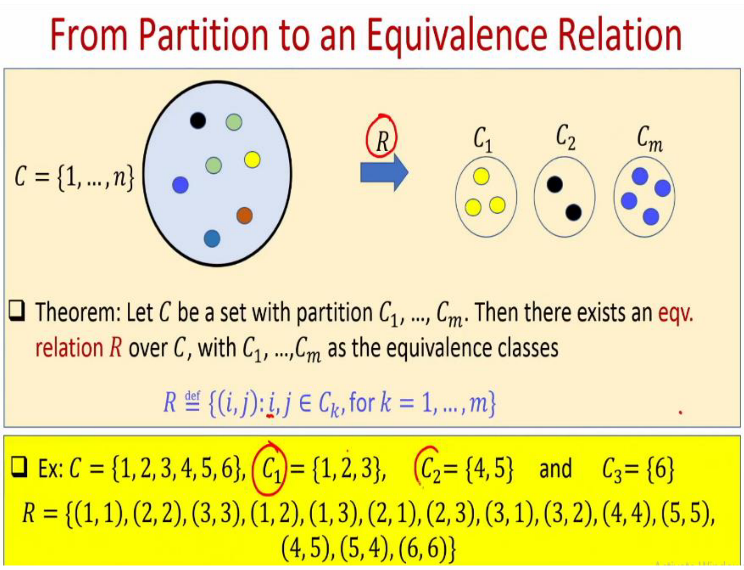

Now, I can prove the property in the reverse direction as well. What do I mean by that? I claim here that you give me any partition of a set C, say you give me a collection of m subsets which constitute a partition of the set C. Then I can give you an equivalence relation R whose equivalence classes will be the subsets, which you have given me in the partition.

So, I will give you the construction of the equivalence relation and the construction of the equivalence relation here is very straight forward. So, the required equivalence relation is the following. You take any subset from the given partition, say the subset $C_k, k = 1, \ldots, m$ because you are given m such subsets in your partition. So, with respect to each subset C_k, what I am going to do is I take $i \in C_k$and $, j \in C_k$, I add the ordered pair (i, j) in my relation R.

So, I stress here that there is no special requirements from my, i and j. I am looping over all possible i, j present in the subset C_k. So, either $i = j$ or $i \neq j$. For every $i \in C_k$and $, j \in C_k$, add (i, j) in my relation R. And if I do this for every subset C_k in my given partition then my claim is that the resultant relation R will be an equivalence relation and its equivalence classes will be the subsets $C_1, \ldots, C_m$.

So, just to demonstrate my point, imagine my set $C = \{1, 2, 3, 4, 5, 6\}$ and a partition of this set is given to you. So, I am given 3 subsets, $C_1 = \{1, 2, 3\}, C_2 = \{4,5\}, C_3 = \{6\}$. Let me construct a relation R as follows. So, I take the first subset here and by iterating over all i, j present in this subset, I add ordered pairs of the form (i, j).

$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2), (4, 4), (5, 5), (4, 5), (5, 4), (6,6)\}$.
So, these are the ordered pairs which I have added with respect to the subsets. With respect to the third subset you might be wondering there is no j present in the third subset. That is why I said there is no restriction that i should be same, i should be different from j or i should be same as j also. So, I have to iterate over all possible i, j present in the subset. So, in this subset if I substitute i = 6 and j = 6, I have to add the ordered pair (6, 6) in my relation R. And now you can check here that the relation R that I have constructed is indeed an equivalence relation it satisfies the reflexive properties. It satisfies the symmetric property and it is transitive as well. And if you form the equivalence classes of this relation R, you will get these three subsets C_1, C_2, C_3.

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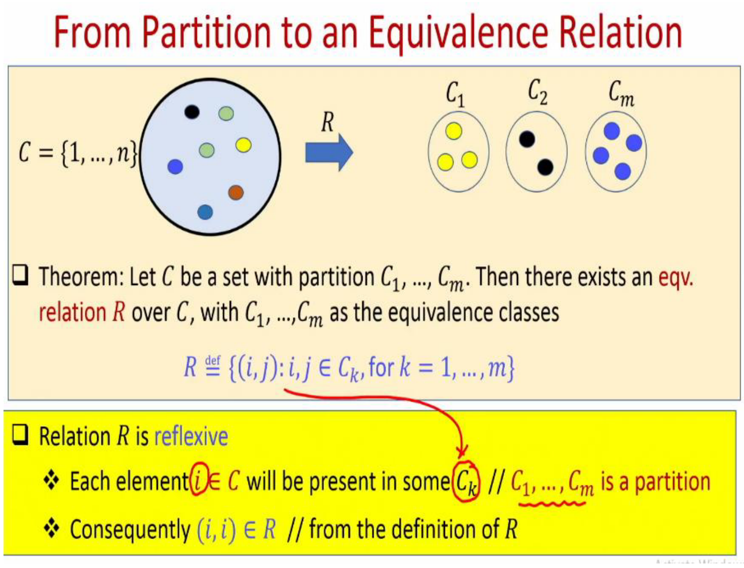

So, let us formally prove this. Now, going to prove that a relation R that I am saying here to construct indeed will be reflexive, symmetric and transitive. So, let us prove that this relation R will be reflexive. So, you take any element i from the set C, I have to show that $(i, i) \in R$ to show that it is reflexive. Now since $C_1, .., C_m$ is a partition of the set C, the element i will be present in one of the subsets in this collection say it is present in the subset C_k.

Now if it is present in the subset C_k when I am applying this rule to construct this relation R, I will see that element i is present in $C_k$and I will add the ordered pair (i, i) in the relation R as per this rule. So, that shows that you take any element $i \in C$, it is guaranteed that $(i, i) \in R.$ That proves that the relation R is reflexive.

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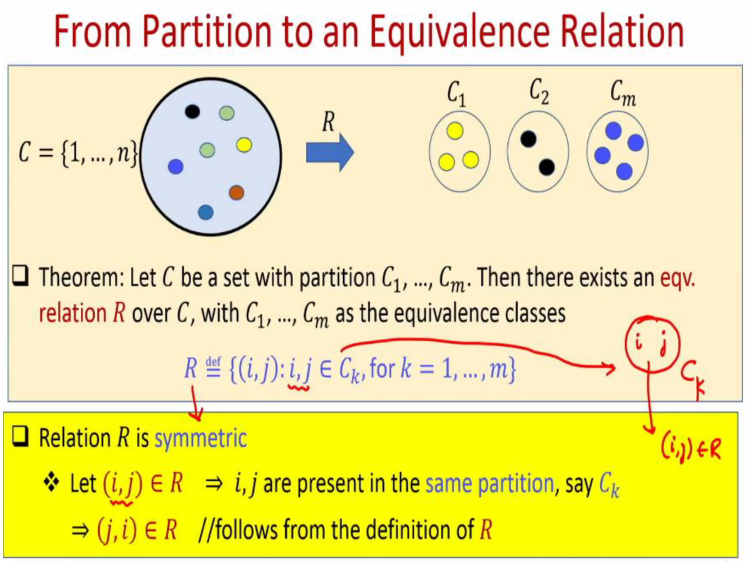

Let us prove that a relation R that we have constructed here is symmetric as well. And for proving that I have to show the following. I have to show that if you take any arbitrary $(i, j) \in R$, then $(j, i) \in R$. And how do I prove that? So, the first thing to observe here is that if at all you have $(i, j) \in R$. That is possible only because of the following.

You have say, $i \in C_k, j \in C_k$, then only you would have added the ordered pair $(i, j) \in R$ and none of these two elements i and j could be present in any other subset in this partition, the given partition or in the given collection of subsets, because that is the definition of a partition. So, since $i \in C_k, j \in C_k$, by applying the rule that I have followed for constructing the relation R, I would have also added the element (j, i), because I have to iterate over all possible i, j. So, when i become j and j becomes i as a result I get $(j, i) \in R$ and that prove that my relation R is symmetric.

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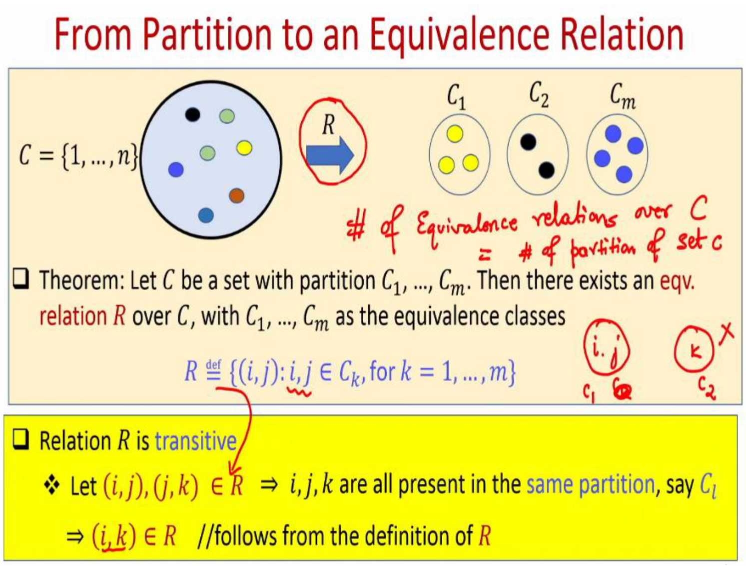

Now let us prove that a relation R is transitive and for proving that my relation R is transitive, let me take an arbitrary ordered pairs. So, I take $(i, j), (j, k) \in R$ and I have to show that the ordered pair $(i, k) \in R$. So, the first thing to observe is that since by construction of my relation R if at all $(i, j), (j, k) \in R$, that is because all the elements i, j, k were present in a common subset namely say subset C_l.

Because it cannot happen that you have $i, j \in C_k$ and you have or say $i, j \in C_l$ and $k \in C_2$. That is not possible here. Because that would have been the case then you would have added the ordered pair (i, j) and (j, i) in the relation but you would have not added the ordered pair (i, k) or (j, k) in your relation. You would have added the ordered pair (i, j) or (j, k) in the relation only when all the three elements $i, j, k \in C_l$.

Now since you would have iterated over all possible $i, j \in C_l$, you would have iterated over k as well and you would have added the ordered pair (i, k) in the relation R as well and that shows that your relation R is transitive. So, that shows a very nice relationship and a nice property between the equivalence classes and the partition.

You give me any equivalence relation the corresponding equivalence classes will constitute a partition. You give me a partition of a set, I will give you an equivalence relation corresponding to those partitions, namely the equivalence relation will be such that its equivalence classes will give you the same subsets which are given in the partition that you given to me.

So, in other words what we can show here is that the number of equivalence relations what we have established here actually is that the number of equivalence relations over C is exactly the same as number of partitions of set C. Because we have established that you give me any equivalence relation that corresponds to a partition you give me any partition that corresponds to an equivalence relation. So, the counting the number of equivalence relations in a sense is same as counting the number of partitions of the sets.

So, that brings me to the end of this lecture. Just to summarize, in this lecture we introduced the notion of partition of a set and we established formally the relationship between an equivalence relation, its equivalence classes and the partition of a set.