Discrete Mathematics
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Module No # 08
Lecture No # 36
Solving Linear Homogeneous Recurrence Equations – Part II

Hello everyone, welcome to this lecture. In this lecture we will continue our discussion regarding how to solve linear homogeneous recurrence equations.

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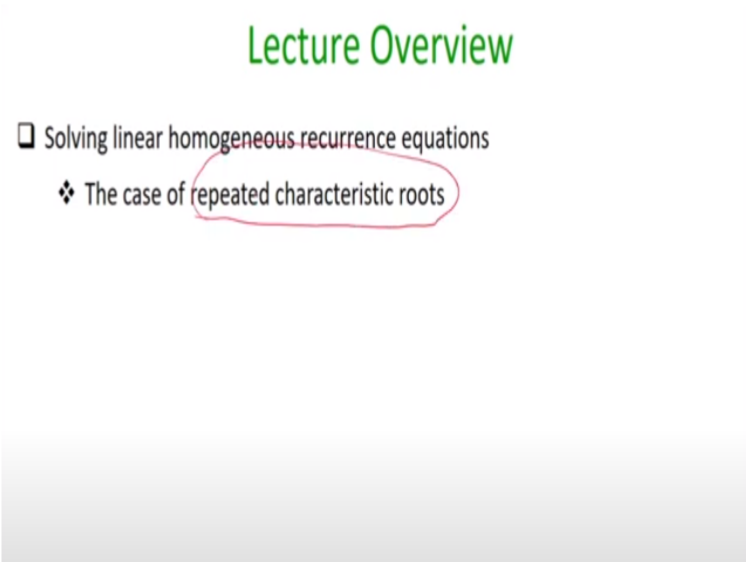

So just to recap, in the last lecture we discussed how to solve linear homogeneous recurrence equations for the case when the characteristic roots were all distinct.

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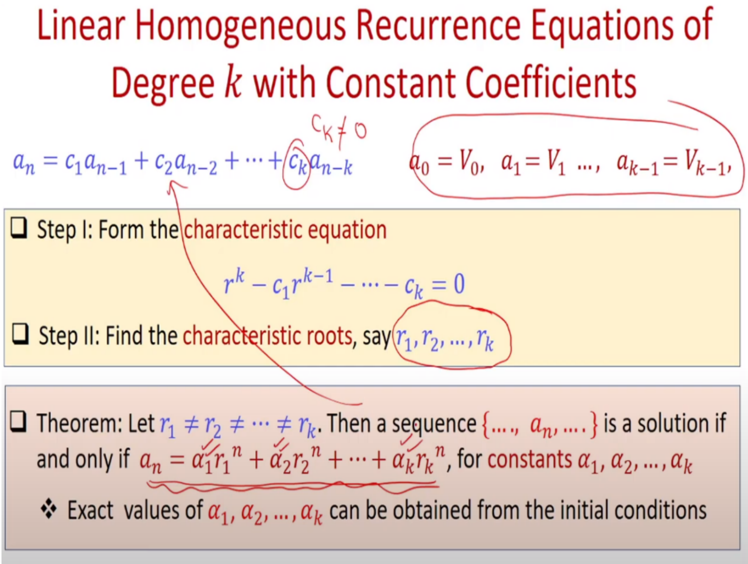

So this was the summary of the discussion that we had in the last lecture. We had a linear homogeneous recurrence equation of degree k, which is imposed by saying that c_k is not allowed to be 0. And you may or may not be given the initial conditions. The first step will be to form the characteristic equation which will be an equation of the degree k and then we find the characteristic roots.

They may be real roots, complex roots, they may be all distinct, some of them may be distinct, some may be repeated and so on. So in total we will have k roots which we denote as r_1 to r_k. And the case that we discussed in the last lecture was the following. If it turns out that all the k roots are different, then any sequence will be satisfying the given recurrence condition provided the n-th term of that sequence is of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$. Namely, it is some constant time the first characteristic root power n plus constant times second characteristic root power n and so on. For some arbitrary constant α_1 to α_k.

Depending upon what are the constant you fit, you get different sequences satisfying the recurrence condition. So this general form of the solution is irrespective of whether it is satisfies the initial condition or not. If you want to satisfy the initial conditions as well, then you can find out the exact constants or the unique solution.

So remember, if we are not bothered about initial conditions then there can be plenty of infinite sequences satisfying the same recurrence condition and all of them can be obtained by this general form. Now if you want to find out the unique sequence satisfying the initial conditions as well then you have to find out the exact values of this constant $\alpha_1, \alpha_2 \dots \alpha_k$ and which can be obtained by utilizing the initial conditions if they are given to you. That is the idea.

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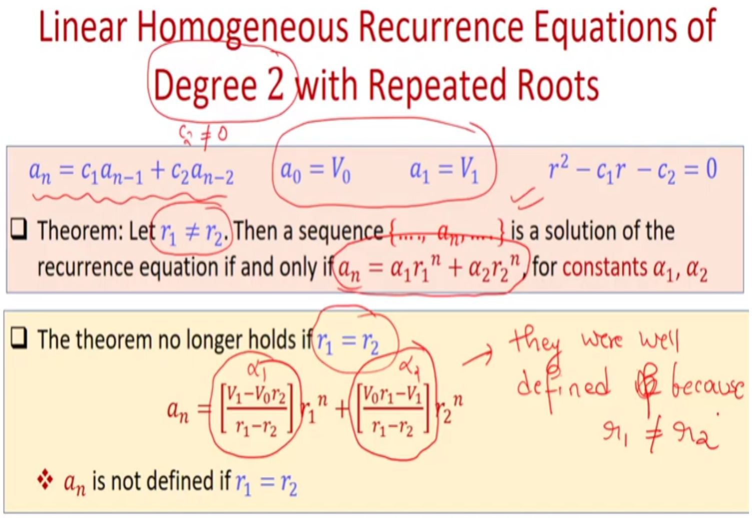

Now, in this lecture we will discuss the case when the characteristic roots are repeated. And again for simplifying the discussion, we start with the case when the degree is 2. That means the general form of the recurrence equation is this, where c_2 will not be 0. And you may or may not be given the initial conditions. So if you are not given the initial conditions then we will end up with the general form of the solution. That means any sequence which will satisfy this recurrence condition what will be the general form for the n-th term of the sequence? We will end up with that.

But if you want to find out the exact sequence then we have to utilize the initial condition as well. So this was the theorem for the case where the roots where distinct. If the roots are distinct then any sequence whose n-th term is of this form for some arbitrary constants α_1 and α_2 will be satisfying the recurrence condition. But it turns out that the theorem no longer holds if the roots are equal.

We are considering the case when we have degree 2 characteristic equation and hence we have 2 roots r_1 and r_2 and if r_1 and r_2 are same then the theorem no longer holds. This is because if you recall the proof of the last theorem then there, if you want find out the exact sequence satisfying the given recurrence condition as well as the initial conditions; so for that our constants α_1 and α_2 turned out to be $\alpha_1 = \left[\frac{V_1 - V_0 r_2}{r_1 - r_2}\right]$ and $\alpha_2 = \left[\frac{V_1 - V_0 r_2}{r_1 - r_2}\right]$.

And they were well defined because for the previous case your r_1 and r_2 were different. In which case the denominator will be non-zero. But if your r_1 and r_2 are same then theseα_1 and α_2 are not well-defined and in which case we cannot find out the exact sequence satisfying the given recurrence condition as well as the initial condition. And if we cannot find out the exact sequence from the general form, that means that general form which was applicable for the previous case is not applicable for this case.

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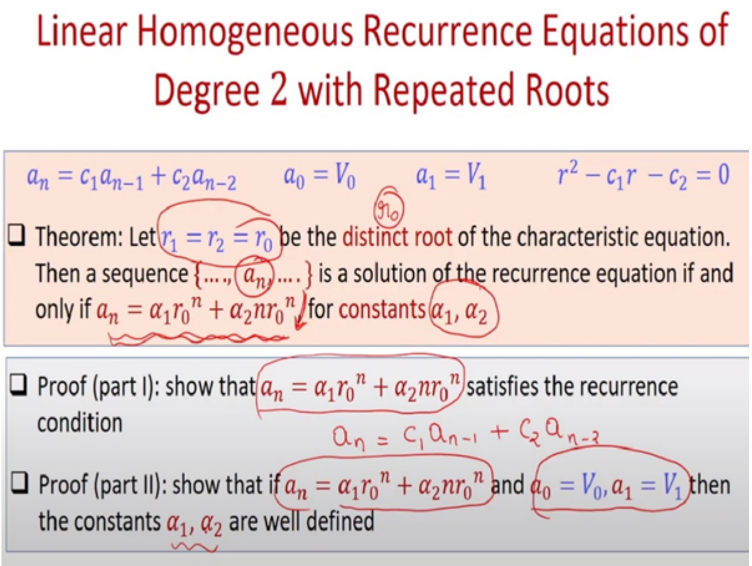

So that means the general form of the solution, or the general form of any sequence satisfying the recurrence condition for the case of distinct characteristic roots will be different. So, the theorem statement for this case is as follows. We can prove that if the characteristic roots are distinct, and say the characteristic roots are r_1 and r_2 are same so I denote the characteristic root as r_0. It is the common characteristic root.

Then this theorem statement basically says that any sequence whose n-th term is of this form will satisfy the recurrence condition for some arbitrary constants α_1 and α_2. Now again, this is the general form of any sequence satisfying the recurrence condition and there could be infinite number of such sequences. So right now I am not utilizing the initial conditions. If you want to find out the exact sequence satisfying the recurrence condition as well as the initial conditions then you have to solve the, or you have to find the exact constants which you can obtain by utilizing the initial conditions, if they are given to you. But if they are not given to you then we end up with the generic form or the general term of any sequence satisfying the recurrence condition and this will be $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ where α_1 and α_2 are constants.

So that is the difference here. In the previous case if r_1 and r_2 are the two roots then the general form was $\alpha_1 r_1^n + \alpha_2 r_2^n$. My claim is that this general form is not applicable for the distinct root case. The general form will be now $\alpha_1 r_0^n + \alpha_2 n r_0^n$. And the proof can be shown similar to the way we proved the theorem for the case of distinct roots.

There we utilize the fact that r_1 and r_2 are 2 distinct root of the characteristic equation and then we showed that whatever general form was there, it satisfies the recurrence condition. The same we have to do in this case as well and we have to utilize the fact that we have equal roots for the quadratic equation. So the part 1 of the proof will be, we have to show that any sequence of the n-th term is this, irrespective of what is the value of the constants α_1 and α_2 will satisfies the recurrence condition that $a_n = c_1 a_{n-1} + c_2 a_{n-1}$. And we can prove that very easily.

And the second part of the proof will be that if at all this sequence whose n-th term is this satisfies the recurrence condition as well as the initial conditions then we can find out the constants α_1 and α_2. They are well defined. Now the form of the constants α_1 and α_2 will be different from the form of the constants α_1 and α_2 which were there for the case of distinct roots.

Now the α_1 and α_2 will not be the same. Specifically, in the denominator, in the earlier case we had $r_1 - r_2$ which was non-zero. But now the form of α_1 and α_2 will be different for this case. And the proof is very similar to the proof for the case of distinct root case. So I am not going into the details you can easily do that.

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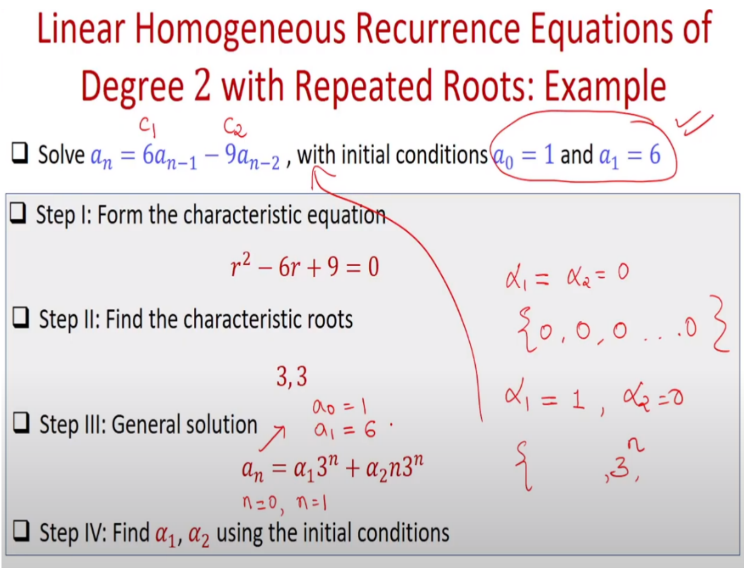

So lets see an example for this case. So suppose I want to solve this recurrence condition and for the moment ignore the initial conditions. So the first step will be forming the characteristic equation, the characteristic equation will be of degree 2. Here $c_1 = 6$ and $c_2 = 9$, so accordingly I get the characteristic equation as $r^2 - 6r + 9 = 0$. Next step will be finding the characteristic roots. And in this case the characteristics roots are same, namely, 3 and 3.

That means the general form of the solution will be $a_n = \alpha_1 3^n + \alpha_2 n 3^n$ where α_1 and α_2 are constants. This will be the general form of the solution. Now substituting the different values of α_1 and α_2 you get various sequences satisfying the recurrence condition. In fact, if you substitute $\alpha_1 = \alpha_2 = 0$ that will give you one sequence. Namely a sequence where all the terms are 0s. That is a sequence satisfying the given recurrence conditions.

If you substitute $\alpha_1 = 1$ and $\alpha_2 = 0$ then that gives you a sequence where the n-th term is 3^n which can be easily verified to satisfy the same recurrence condition. So, by substituting the different values of α_1 and α_2 you will get different sequence satisfying the recurrence condition.

But now since we are also given the initial conditions we will be interested to find out the exact sequence satisfying the recurrence condition as well as having the initial terms 1 and 6. So for that, I will be substituting $n = 0$ here and $n = 1$ here and get $a_0 = 1$, $a_1 = 6$ which will give me now 2 equations in α_1 and α_2 where α_1 and α_2 are my unknown constants and by solving those two equations I will be find out the exact α_1 and α_2 and hence the exact sequence satisfying my recurrence condition as well as the initial conditions.

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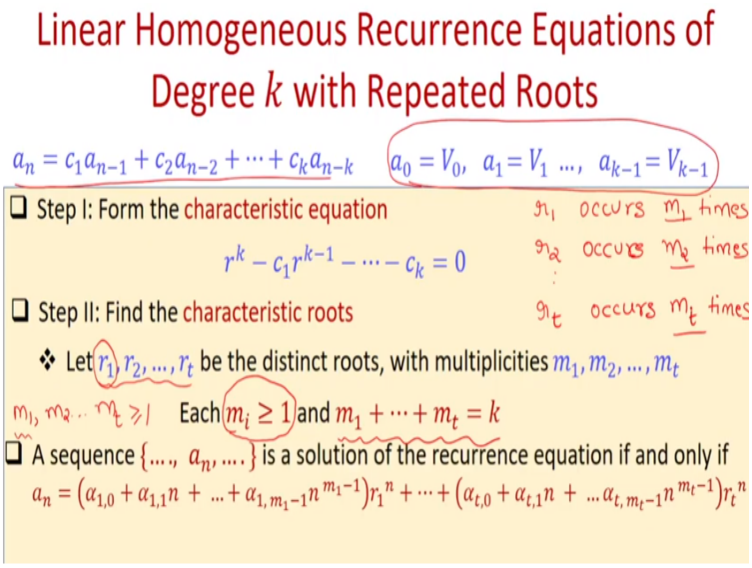

So that was the case of degree 2 characteristic equations with repeated roots. Now we will consider the general case where we have a linear homogeneous recurrence equation of degree k with repeated roots. So, this will be the form of the recurrence equation. You may or may not be given the initial conditions if you are not given the initial conditions, we will just end up writing down the general term of the sequence satisfying the recurrence condition.

But if you are also given the initial conditions you can find out the exact sequence satisfying the initial conditions as well as the recurrence condition. So, the first step here will be forming the characteristic equation which will be an equation of degree k and hence it will have k characteristic roots. Some of them might be distinct, some of them might be same, and so on. So what I do here is, I denote the distinct roots by r_1 to r_t and they occurred with multiplicities m_1 to m_t respectively.

That means the root r_1 occurs m_1 times, the root r_2 occurs m_2 times and like that the root r_t occurs m_t times. What is the relationship here? Each of these values m_1, m_2 and m_t are greater than equal to 1, because these are the distinct roots. So r_1 will be occurring as a root definitely at least once. So that is why m_1 will be at least l. Of course, it might be possible I have 2 roots, namely, r_1 and r_1 in which case m_1 will be 2; or I might have a case when the only root is r_1. That means in that case m_1 will be k and so on.

That means depending upon number of times r_1 is occurring as a root determines the value of m_1. Since r_1 is occurring as a root definitely at least once, m_1 will be at least once. Similarly m_2 will be at least 1 and similarlym_t also will also be at least 1. So, that justifies why each m_i is greater than equal to 1. And the second condition here is that the sum of all the multiplicities here will be k because you have total k number of roots.

So r_1 will be occurring certain number of times as root, r_2 is occurring certain number of times as a root, and r_t is also occurring as a root certain number of times, namely m_t number of times. If I sum up all these multiplicities, that should give me the total number of roots, namely, k. That is the relationship between these multiplicities and the various roots. Now what the theorem basically says is the following.

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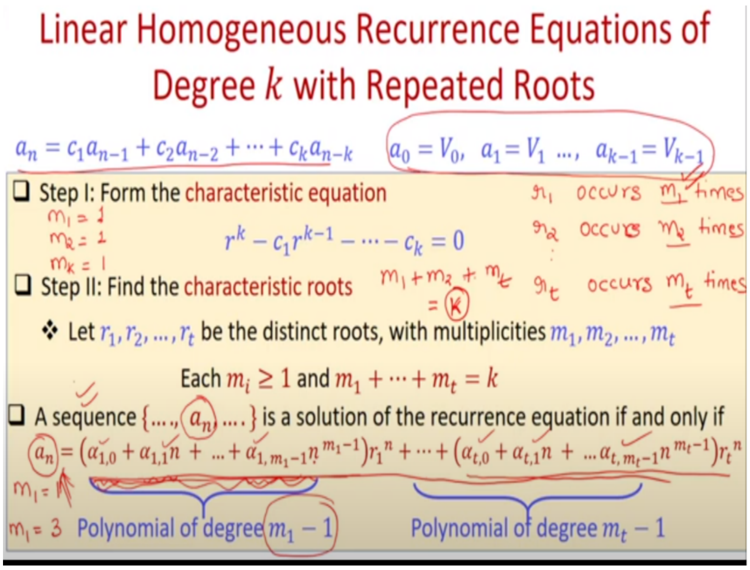

The theorem says that any sequence whose n-th term is of this form will satisfy the recurrence condition.

So let us decode this complicated looking general term here. So, what is the first term here? The first term is basically a polynomial of degree $(m_1 - 1)$ multiplied by the first characteristic root raised to power n. Why $(m_1 - 1)$? Because the first characteristic root namely r_1 occurs m_1 number of times. So that is why a polynomial degree $(m_1 - 1)$. So that means if r_1 occurs as a root only once, in that case m_1 would have been 1.

In which case, this polynomial of degree $(m_1 - 1)$ is nothing but a constant. So, in that case it will be constant times r_1^n. But if m_1 is say 3, that means r_1 occurs as a root 3 number of times, then this polynomial, the first polynomial here will be a polynomial of a degree 2 multiplied by the first characteristic root power n. The constant, the coefficients in this polynomial of degree $(m_1 - 1)$ they are all unknown constant.

We can substitute any value for them and that will determine the sequence satisfying the recurrence condition. The next term in this general term will be an unknown polynomial of degree $(m_2 - 1)$ multiplied by the second characteristic root power n. And like that the last term will be a polynomial of degree $(m_t - 1)$ because the last characteristic root namely r_t has multiplicity m_t.

So it will be raised to power n and it will be proceeded by a polynomial of degree $(m_t - 1)$. So all these alpha values, they are constants here, unknown constants. By substituting various values for this alphas you get various sequences satisfying the recurrence condition. Now if you want to satisfy the initial conditions as well, then you can substitute the values of $n = 0, n = 1$ and all the way $n = k - 1$.

You get various equations in this unknowns alphas. And you can get the exact alphas satisfying the initial conditions as well as the recurrence condition as well. Specifically, how many alphas will be there? In the first polynomial you have m_1 alphas. Second polynomial you had m_2 alphas and like that the last polynomial you had m_t alphas. So total how many unknowns are there? $m_1 + m_2 + \cdots + m_t$ which is nothing but k.

So you will be having k unknown constants here and if you are given k initial conditions namely V_0, V_1 up to V_{k-1} by substituting $n = 0, n = 1, \ldots, n = k - 1$ in this general formula you get k equations in k unknowns, by solving them you get the exact constants. Now you can tell that this general formula captures all the cases that we had discussed still now.

So if you are in the case when all the roots are distinct, that means you have k roots, each of the roots has multiplicity 1. That means m_1 is 1, that means r_1 occurs as a root exactly once. m_2 is 1, that means r_2 occurs as a root exactly 1 and like that m_k is 1 that means r_t occurs as a root exactly once. Then each of these polynomials will be a constant polynomial followed by that characteristic root raised to the power n. And that is precisely was the general form of a sequence or the n-th term of the sequence satisfying the recurrence condition for the case when all the characteristic roots where distinct.

So you can easily verify that. The proof for this theorem will be again similar to the case where we took the degree 2 equation and the roots where distinct. I am leaving the proof for you.

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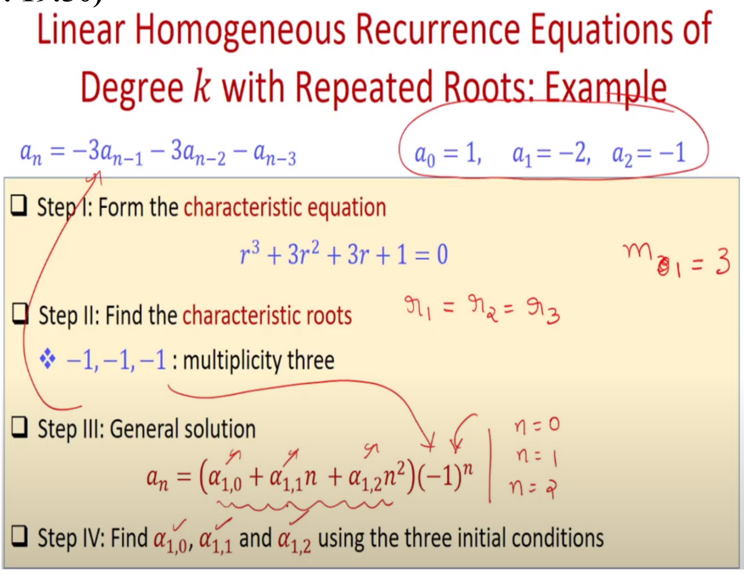

So now let see an example for applying the general formula. So imagine you are given the recurrence condition $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ and for the moment assume you are not given the initial conditions. So the first step will be forming the characteristic equation. The characteristic equation will be $r^3 + 3r^2 + 3r + 1 = 0$. It is a cubic equation so it will have three characteristic roots r_1, r_2, r_3. It turns out in this case that r_1 and r_2 and r_3 are all same. And hence the number of roots is 1 and its multiplicity is 3.

So the general solution will be a known polynomial of degree $m_1 - 1 = 2$, multiplied by the characteristic root raised to the power n and the characteristic root here is -1. Thus, the general solution is $a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)(-1)^n$. By substituting different values of the$\alpha_{1,0}, \alpha_{1,1}$and $\alpha_{1,2}$ you get various sequences satisfying the recurrence condition.

Now since you are also given the initial condition you can find out the exact sequence satisfying this initial condition. That means you start with the terms 1, -2, -1 and n-th term satisfies the recurrence condition. That you can do by substituting $n = 0, n = 1$, and $n = 2$ in general form and equating the resulting expression with 1, - 2, and -1 respectively. This results in 3 equations which can in turn be used to solve for $\alpha_{1,0}$, $\alpha_{1,1}$ and $\alpha_{1,2}$ and you get the exact value of this constants and hence the exact sequence satisfying the recurrence condition.

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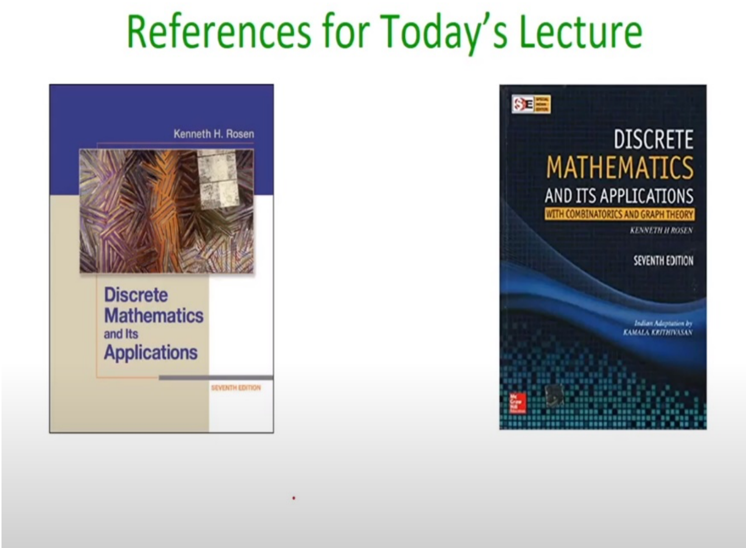

So that brings to me to the end of today's lecture. So these are the references used for today's lecture. Just to summarize, in this lecture we continued our discussion regarding how to solve linear homogeneous recurrence equations of degree k and we consider the case when the characteristic roots are repeated. Thank you.