Discrete Mathematics
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Lecture -20
Tutorial 3

Hello everyone welcome to tutorial number 3.

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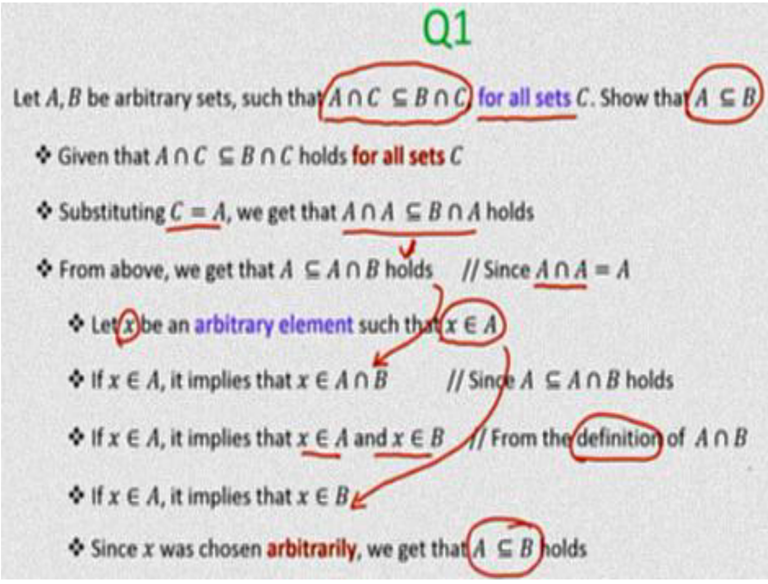

So, let us start the question number 1, here you are given that arbitrary sets A and B. And the sets A and B as such that, this condition holds namely $(A \cap C) \subseteq (B \cap C)$ for any set C that you consider. If that is the case and you have to show that $A \subseteq B$, so you are given the premise that the $(A \cap C) \subseteq (B \cap C)$ for any set C.

So, since this condition holds for any set C if I substitute C = A in this condition then I get that $A \cap A \subseteq B \cap A$, but I know that $A \cap A$ is nothing but the set A, so that means I can say that my premise, which I obtained by substituting C = A is that is $A \subseteq A \cap B$.

Now my goal is to show that $A \subseteq B$, so for showing that $A \subseteq B$; I have to show that you take any element x in the set A, it should be present in the set B as well. So, I am taking an arbitrary element x and I am assuming it is present in the set A and since, I have established the premise that $A \subseteq (A \cap B)$, that means if since the element x is present in A same element x will be present in the $A \cap B$ as well.

Now, as per the definition of A ∩ B it means the element x is present in A and simultaneously it is present in B as well, that means definitely it is present in B as well, so what I have established here is that if I start with an arbitrary element x which is present in the set A, I have established that it is present in the set B as well where and since x was chosen arbitrarily here.

This is true for any element x that you choose from the set A and hence I get the conclusion that A ⊆ B.

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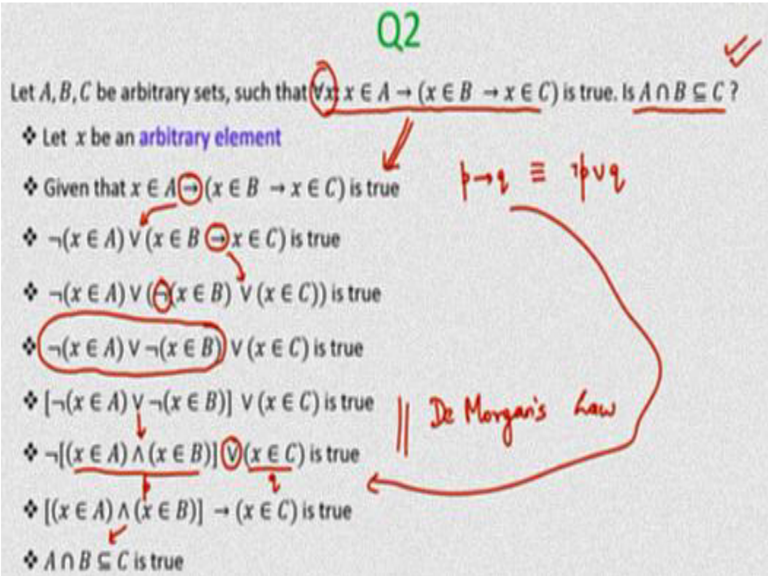

In question two you are given the following; you are given 3 sets A, B, C such that this predicate holds for every element x in the set A and the property here is that if x ∈ A then the implication that x ∈ B → x ∈ C is true and this is a universally quantified statement, that means this condition holds for every element x in the set A.

Then you have to show, you have to either prove or disprove whether the A ∩ B ⊆ C or not, so in fact we are going to prove this statement we will prove that you take any element x which is arbitrarily chosen, and if it is present in the A ∩ B then it is present in C as well. So, we start with an arbitrary we chosen an element x and we start by reworking the premise that is given to be true here.

So, the premise here is that this universally quantified statement is true for every element x. So, what I am doing here is I am rewriting this implication here, so remember the statement p → q is logically equivalent to the disjunction of negation p and q. So, that is why I have splitted this implication into a disjunction and I get this equivalent form, then I apply the same rule again over this implication.

So, this implication is now replaced by this disjunction and I put a negation here. Now you see everywhere I have disjunction, I can apply the associativity property of the disjunction. I can club together the disjunction of the first two statements here. Now what I can say is that, the disjunction of these two negations is equal to the negation of the conjunction of these two statements, this is from the De Morgan's law.

And again I can use the same rule that I have used here to replace this ¬p ∨ q by p → q, so this you can interpret as p this you can interpret as q, so you are given ¬p ∨ q; I can rewrite it as p → q and now what is this condition? This condition is nothing but it says that if x is present in A and if x is present in B it implies x is present in C.

So, since x was arbitrarily chosen and that means this condition holds for any x from the set A and the condition that x belongs to A and x belongs to B simultaneously means x belongs to the A ∩ B and since x belongs to C as well, by the definition of subset it follows that the A ∩ B ⊆ C. So, this statement is a true statement.

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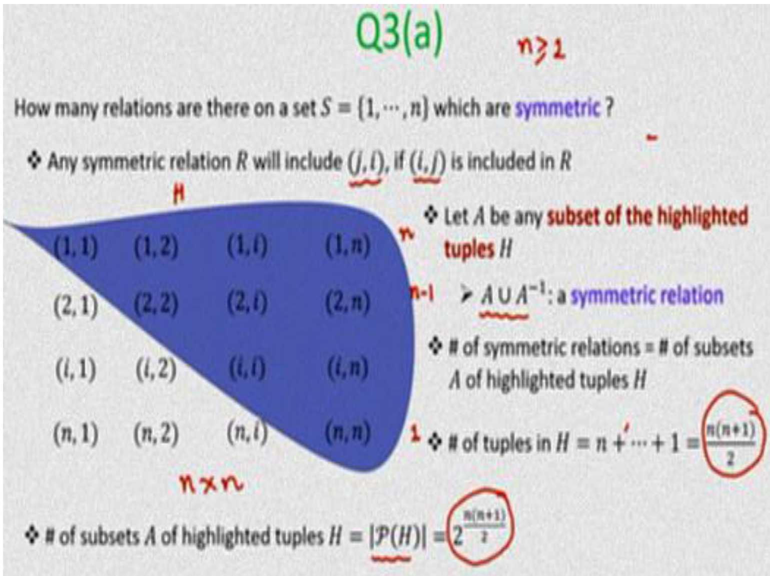

Now in question 3, there are several parts and, you are given a set S, consisting of n elements where n is not 0. So, implicitly I am assuming here that n is greater than equal to 1. And now I have to count various number possible relations satisfying some properties. So, the first property here is I am interested to count the number of relations on this set S which are symmetric.

And just to recall that definition of a symmetric relation is that if you have the element or the ordered pair (i, j) present in the relation R then the ordered pair (j, i) should also be present in the relation R that is the requirement for a symmetric relation. So, what I have done here is I have drawn all possible n^2 ordered pairs, these are the ordered pairs in the set S x S.

So, you can have n^2 such ordered pairs and any subset of these n^2 ordered pairs will constitute a relation over the set S. I have to find out how many ways I can take subsets of this n^2 ordered pairs such that resultant subset satisfies the property of a symmetric relation. So, now among this n^2 ordered pairs, I have highlighted this selected ordered pairs.

So, you can imagine this matrix this is n x n matrix and I am focusing only and the upper triangular quotient of this matrix and let A be any subset of the highlighted ordered pairs. So, the highlighted tuples, I am calling it as H and any subset of that, I am calling it as A, my claim is that if you take the union of the ordered pairs in the subset A and the ordered pairs in A^{-1} then the collection of these ordered pairs will give you a symmetric relation.

So, say for instance, if you pick A is equal to say (1, 1) and say (2, 3), say this is a subset of H, then A^{-1} will be the inverse of this ordered pairs inverse means taken by reversing or swapping the order of the ordered pairs. So, (1, 1) its inverse will be (1, 1) but the inverse of (2, 3) will be (3, 2) and now if I take the union of these two relations that will give me a symmetric relation.

So, like that, I take any subset A of the highlighted tuples and to that I add all the ordered pairs which are found by taking the inverse of the ordered pairs in the set A, then the collection will be a symmetric relation and that is the only way you can construct a symmetric relation over the set S because the requirement for a symmetric relation is that if (i, j) is there if you have included (i, j) then you have to include (j, i) as well.

So, what I am intuitively saying is that in the set A you first decide what are the (i, j) pairs that you are going to include. once you have decided how many (i, j) pairs you are going to includes just take the inverse of those (i, j) pairs and the resultant union will give you a symmetric relation. Now how many ways you can fix your, (i, j) pairs well those (i, j) pairs I am asking you to pick from the upper triangular portion of this matrix.

Once you have fixed which (i, j) pairs you are going to select from this upper triangular matrix just take the inverse of those (i, j) pairs from the lower portion of the matrix, and that will give you a symmetric relation. That means I can say here that the number of symmetric relations that I can form is nothing but the number of subsets A that I can choose from the highlighted tuples.

So, how many elements are there in the highlighted tuples? So, my claim is it is n * n + 1) / 2 this is, because in the first row you have n number of highlighted, n number of tuples, in the second row of the upper triangular matrix you have n - 1 number of ordered pairs and like that, in the last row you have only 1 ordered pair say if I sum these things I get n * (n + 1) / 2 number of ordered pairs.

And how many subsets of these ordered pairs I can form? That is nothing but a cardinality of the power set of the tuples in your set H and since the set H has these many number of elements namely n * (n + 1) / 2, the number of subsets of these order tuples that I can form is nothing but $2^{\frac{n*(n+1)}{2}}$. So, that is the number of symmetric relations that you can form over a set S consisting of n elements.

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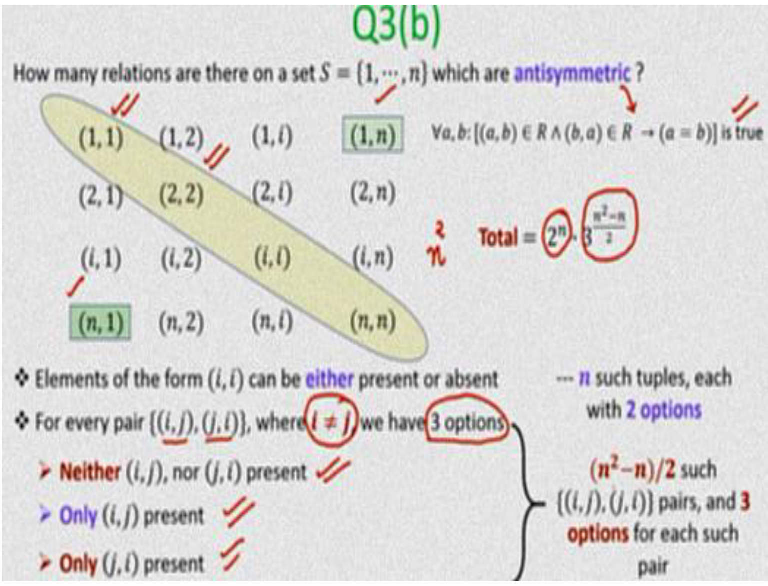

Now in part b of the question I want to find out the number of relations over the set S, which are anti-symmetric and just to recap, this is the requirement from a anti-symmetric relation. If both (a, b) and (b, a) are present in the relation then that is allowed only if a is equal to b that means a is not equal to b contra-positively if a is not equal to b then you cannot have simultaneously both (a, b) as well as (b, a) present in the relation R.

So, now count the number of anti-symmetric relations I do the similar thing here I have drawn all the n^2 possible ordered pairs. Now when you are trying to form an anti-symmetric relation the first thing to observe here is that the ordered pairs along the diagonal here can be either present or absent in your anti-symmetric relation that means if say for instance (1, 1) and (2, 2) are included in your relation then that does not violates the requirement of anti-symmetric property.

Because if (a, b) and (b, a) simultaneously present then that is allowed only when a is equal to b and all the ordered pairs along the diagonal elements have a is equal to b. Mind it, it is not mandatory that all the n ordered pairs along the diagonals should be present in an anti-symmetric relation, even if none of them is present that is fine that still satisfies the requirement of an anti-symmetric relation.

The condition is that, if at all both (a, b) and (b, a) are there then a equal to b and nowhere, it says that for all (a, b) both (a, b) and (b, a) it should be present in your relation for a equal to b. So, now how many ordered pairs I have along the diagonal? I have n such ordered pairs and for each ordered pair I have 2 option either to include it or to exclude it in an anti symmetric relation.

Now let us focus on the non diagonal ordered pairs of the form (i, j) and (j, i) where i and j are distinct, for instance say (n, 1) and (1, n). Now how many ways I can include these ordered pairs (1, n) and (n, 1) and still satisfy the requirements of an anti symmetric relation. I have 3 possibilities here, possibility 1 that neither (i, j) nor (j, i) is included in my relation R.

So, for instance, I do not include (1, n) and I do not include (n, 1) in my relation that is fine that still satisfies the requirement of an anti-symmetric relation. Option number two; that I include only (i, j) in the relation but exclude (j, i) remember I cannot have both (i, j) as well as (j, i) in my relation, because I am considering the case when i is different from j and if i is different from j then as per the property of an anti symmetric relation both ordered pairs (i, j) and (j, i) are not allowed to be included.

So, I have only three possibilities include none of them or include (i, j) or include (j, i). Now how many such (i, j) and (j, i) pairs I have in this matrix where i and j are different well I have $(n^2 – n)/2$ number of such (i, j) and (j, i) pairs this is because there are total n^2 ordered pairs and from there, I am excluding the elements along the diagonals.

So, I am left with n^2 - n number of pairs, n^2 - n number of elements, and if I pair them in the form (i, j) and (j, i) then I have to divide it by 2. So, for every such pair I have 3 options and the possibility of including or excluding (i, j), (j, i) pairs is independent of the possibility of including or excluding the ordered pairs along the diagonal element when I am forming an anti-symmetric relation.

So, that is why the total number of ways of forming an anti-symmetric relation is how many ways I can consider the elements along the diagonal which is 2^n and how many ways I can consider the remaining elements and that is $3^{\frac{n^2-n}{2}}$, and if I multiply them that will give me the total number of relations which can be anti-symmetric.

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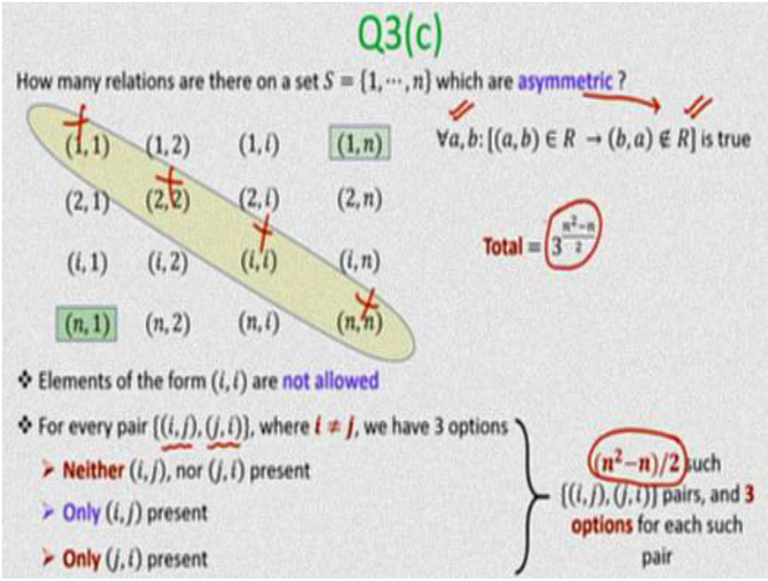

In part C of third question, you are supposed to find out the number of relations which are asymmetric and this is the requirement for asymmetric relation. If at all (a, b) is there in your relation then (b, a) is not allowed in the relation. This does not mean that, that for every (a, b) you should have either (a, b) or (b, a) present in the relation, it is fine if none of them is there in your relation or not.

So, again, I do the same thing here, I consider n^2 ordered pairs and see how many ways I can select subset of these n^2 ordered pairs and still satisfy this universal quantification. So, the first thing to observe here is that now none of the elements or the ordered pairs along the diagonal are allowed in an asymmetric relation.

Because, if you include (i, i) then that violates this universal quantification because you have both (a, b) as well as (b, a) present in your relation that is not allowed. So, none of these, so (1, 1), (2, 2), (i, i) (n, n) none of them are allowed in an asymmetric relation. Now, let us focus on the remaining elements and again, let us club them into pairs of the form (i, j) and (j, i) where i and j are distinct and again it is easy to see that for such (i, j) and (j, i) pairs I have three possibilities here.

I can include none of them and still my relation will be asymmetric. I can include (i, j) and exclude (j, i) or I include (j, i) and exclude (i, j). So, I have three possibilities here and how many such pairs, you can have $(n^2 – n)/2$ such pairs. So, as a result since I have to definitely exclude the diagonal elements the only possibilities I have now is to consider the non diagonal elements.

And, with respect to non diagonal elements I have these many options. So, these are the total number of asymmetric relations.

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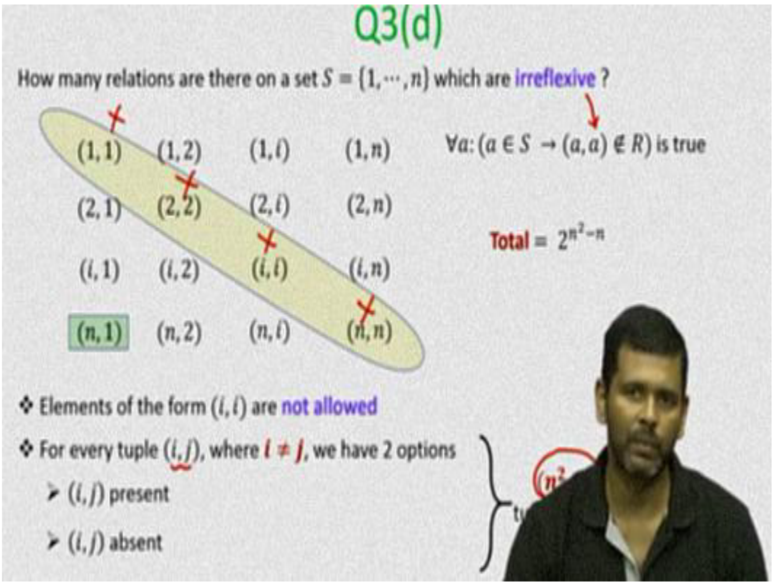

In part d, I am interested to find out how many relations I can form which are irreflexive and the definition of a irreflexive relation is this, it states that for every element a in the set S (a, a) should not be there in your relation R. That means none of the elements none of the ordered pairs along the diagonal are allowed because that will violate this universal quantification.

Whereas if I take any other tuple (i, j) where i and j are different then I can either include it or exclude it that would not violate the requirement from an irreflexive relation. So, for instance if (n, 1) is present in my relation that is fine, that satisfies this universal quantification and I can have (n, 1) as well as (1, n) and both present here that still satisfies the requirement for this form of irreflexive relation.

So, how many such (i, j) ordered pairs I have here, if I exclude the diagonal elements I am left with these many ordered pairs and for each such ordered pair I have two possibilities either include it or exclude it. So, that is why the total number of your reflexive relations is $2^{n^2 - n}$.

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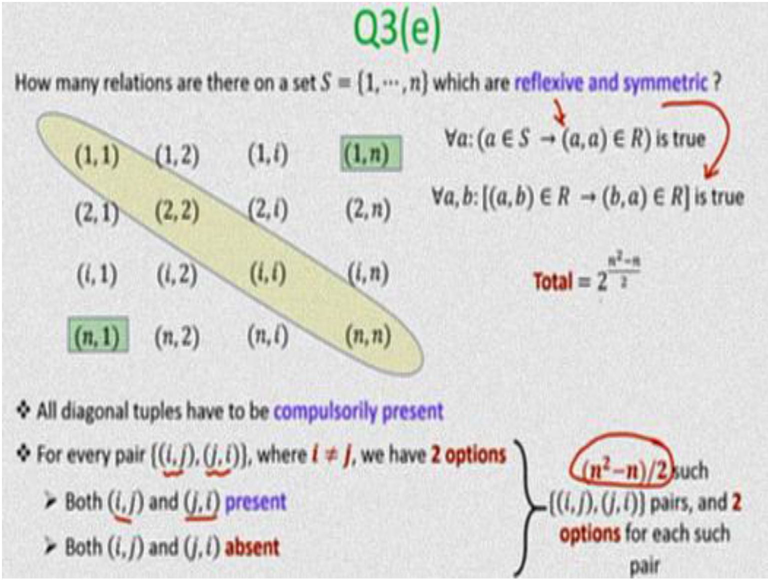

Part e, I now want to find the number of relations which are simultaneously reflexive as well as symmetric and the requirement from a reflexive relation is this and requirement from a symmetric relation is this. So, if I consider the ordered pairs I have no choice, but I have to definitely include all the diagonal ordered pairs because I have to satisfy the reflexive property.

But apart from this, apart from the ordered pairs along the diagonal elements I have the choice with respect to the elements of the ordered pairs which are not there along them. So, I consider all (i, j) and (j, i) pairs where i and j are different. So, I have now two possibilities; I can choose to exclude both (i, j) as well as (j, i) that still satisfies the requirement of symmetric relation or if I decide to include (i, j) then I am forced to include (j, i).

Because, if I include only (i, j) but exclude (j, i) then that will violate the requirement of a symmetric relation. So, with all such ordered pairs of the form (i, j) and (j, i) where i and j are different I have two possibilities and how many such (i, j) and (j, i) pairs are there, where i and j are distinct I have $(n^2 – n) / 2$ such pairs and since I have no option, no choice with respect to the diagonal elements I am forced to include them.

The total number of relations which are both reflexive and asymmetric is nothing, but $2^{\frac{n^2-n}{2}}$.

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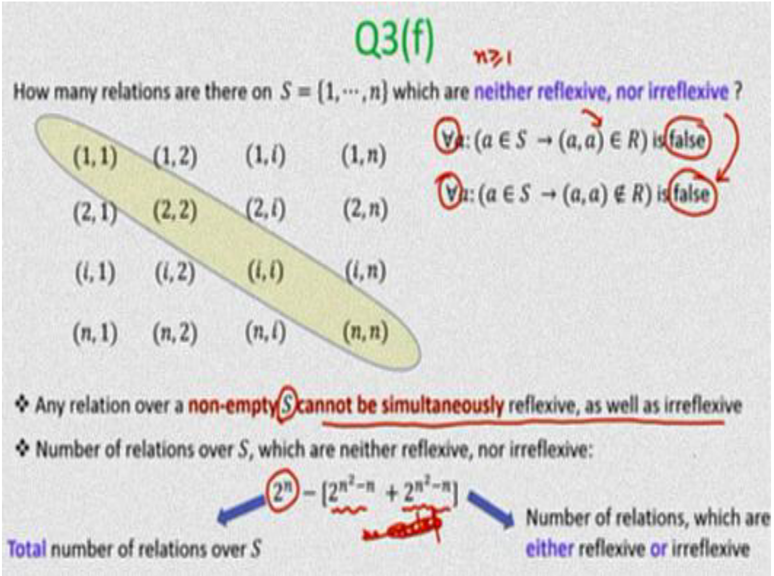

The last part of the question, I have to find out the number of relations, which are neither reflexive nor irreflexive. So, not reflexive means this universal quantification should be false that means at least 1 element a, should be there in the set S such that (a, a) is not present in the relation, then only this universal quantification can become false.

And, not irreflexive means, this second universal quantification is false, that means you have at least one element a in the set S such that (a, a) is present in your relation then only this second universal quantification can be false. So, if I consider this n^2 ordered pairs, then what I can say here is that, since my set S is non empty, so I stress I assumed here n is greater than equal to 1.

So, since my set S is non empty, I cannot have a relation which is simultaneously reflexive as well as irreflexive, right? It cannot satisfy both, it cannot satisfy both the requirements of a reflexive relation as well as irreflexive relation. Because, reflexive relation says that you should have all the elements along the diagonal present in the relation, whereas the irreflexive property demands that none of the ordered pairs along the diagonal should be present.

So, you cannot have both these conditions occurring simultaneously in the relation R. So, I have to exclude all the elements along the diagonal. It turns out that if I try to find out the number of reflexive and your reflexive relation simultaneously with respect to the options that I have for the non diagonal elements then the counting might become slightly tricky.

So, instead what I do here is, I find out the number of relations which are either reflexive, or irreflexive and subtract it from the total number of relations, which I can form over the set S. What I know here is since I cannot have a relation which is simultaneously reflexive as well as irreflexive then I can confidently say that, that if I subtract out the total number of relations which are either reflexive or irreflexive from the total number of relations that will give me the required answer.

Because there will be no overlap which is possible here, no overlap in the sense there cannot be any relation which is simultaneously reflexive as well as irreflexive. So, my goal here was to find out the relations which violate the requirements of being reflexive and irreflexive. So, to do that what I do is I find out the relations which have one of these two properties.

That means, it is either reflexive or irreflexive and subtract out the number of set of relations from the set of all possible relations, which I can have over the set S. The total number of relations, which I can have over the set S is 2^n and if I subtract out the number of reflexive relations which is 2^{n^2-n} which we have calculated.

And the number of irreflexive relations which also is 2^{n^2-n} and that will give me the total number of relations which are neither reflexive nor irreflexive.

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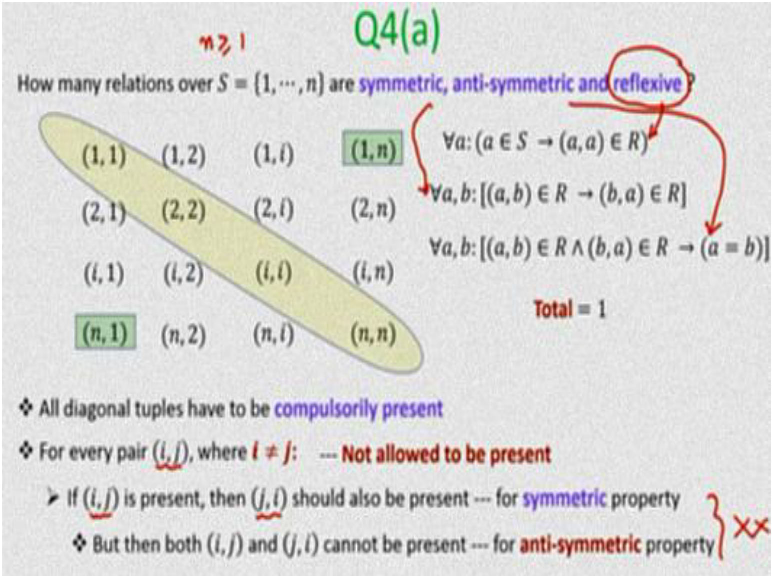

Now, let us start question 4, again, we have several parts and here again, we are given a set S consisting of an elements and I assume here n is greater than equal to 1. I have to find out the number of relations which are symmetric, anti-symmetric and reflexive. So, these are the requirements from my relation, my relation should satisfy the property of a reflexive relation.

My relation should satisfy the property of a symmetric relation and my relation should satisfy the property of an anti-symmetric relation as well. So, I consider the ordered pairs, n^2 ordered pairs. So, since I want my relation to be reflexive, I have no choice but I have to compulsorily include all the ordered pairs along the diagonal here. Now, I consider ordered pairs of the form (i, j) where i is different from j and try to see what I can do with this ordered pairs.

So, that my resultant relation is both symmetric as well as anti symmetric. So, in order to maintain the symmetric property if I decide to include the ordered pair (i, j) in my relation, then I need to include (j, i) as well. But as soon as I include (i, j) and (j, i) in my relation, it will violate the requirement of an anti symmetric relation because anti-symmetric relation says that if your, i and j are different then both (i, j) and (j, i) cannot be present in the relation.

That means these 2 conditions cannot occur simultaneously; that means, if I have ordered pairs of the form (i, j) where i is different from j, I am not allowed to include such ordered pairs because, that will violate the requirement of symmetric and simultaneously anti-symmetric property. So, that means I have now only one way of forming the relation.

Namely, I just include all the ordered pairs along the diagonal and that is all, that relation will be symmetric, anti-symmetric and reflexive apart from that I can not do anything else I can not include any other ordered pair. So, that is why the total number of relations which satisfy these three properties simultaneously is 1.

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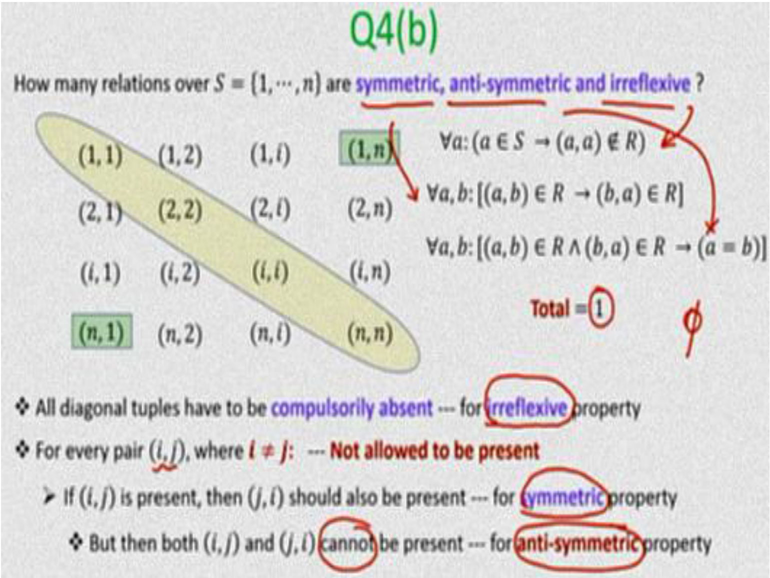

Part b, I have to find out the number of relations which are symmetric, anti-symmetric and irreflexive. So, let me write down the requirements; irreflexive means no element, no ordered pairs of the form (a,a) should be there. Symmetric means, if (a, b) is included and (b, a) should also be there and anti-symmetric means, if you have distinct a and b then either (a, b) or (b, a) should be there not both of them, it is fine if none of them are there.

So, again, I consider the n^2 possible ordered pairs here. Now since my relation has to be irreflexive, I have no choice but I have to exclude, compulsorily exclude all the ordered pairs along the diagonal elements, all along the diagonal. Now, what about the ordered pairs of the form (i, j) where i and j are different. So, again, if I want to include (i, j) then to retain the symmetric property I have to include (j, i).

But since, i and j are different if I have both (i, j) and (j, i) present in my relation then it will violate the requirement from an anti-symmetric relation, that means again here with respect to the ordered pairs (i, j) where i and j are different I am not allowed to have the ordered pair (i, j) because if I try to include (i, j) then it will either violate the requirement of symmetric property or anti symmetric property.

And since the diagonal elements are also not allowed to be included it turns out that only relation which I can form which is simultaneously symmetric, anti-symmetric as well as irreflexive is the empty relation namely include no ordered pair; that means none of the ordered pairs among these n^2 ordered pairs should be there in my relation. Then only my relation can have all the three properties simultaneously and that is why there is only one relation possible.

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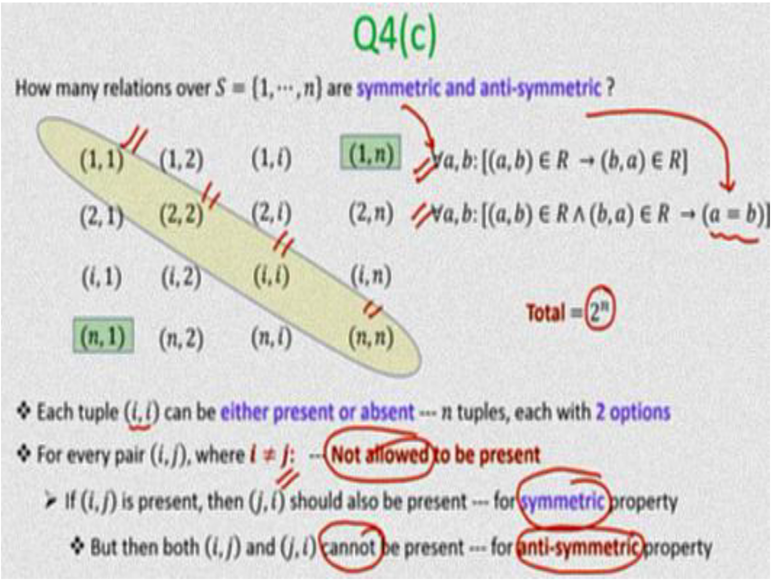

Part c of the question, I have to find out the number of relations which are both symmetric and anti-symmetric. So, this is the requirement from a symmetric relation and this is the requirement from an anti_symmetric relation. So, now if I consider the elements, the ordered pairs along the diagonal, for each ordered pair (i, i); I can either decide I can either choose to include it in my relation or exclude it in the relation, that will satisfy both this universal quantification.

That means, if I say to decide only to include (1, 1) in my relation that is fine that does not violate the requirement from a symmetric relation or it does not violate the requirement from an anti-symmetric relation, because anti-symmetric says that if a and b are same then fine you can have both (a,b) and (b,a) present in the relation and mind it, I am not enforced to have all the ordered pairs along the diagonal to be included in my relation, I can just select any subset of them.

What about the ordered pairs of the form (i, j)? Where, i and j are distinct. Again, if I decide to include (i, j); I am forced to include (j, i) to retain, maintain the symmetric property, but since i and j are distinct if I cannot have both (i, j) and (j, i) simultaneously present in my relation because that will violate the anti-symmetric property. So, again, the conclusion here is that for every ordered pair of the form (i, j) where i and j are different, I am not allowed to include them.

That means, the only options I have is with respect to the diagonal ordered pairs and I can take any subset of the ordered pairs along the diagonal that will satisfy the requirements of a symmetric and anti-symmetric relation and how many such subsets I can form, I can form 2^n subsets. So, this will be the number of relations which will satisfy simultaneously symmetric as well as anti-symmetric. So, with that we end our tutorial number 3. Thank you.