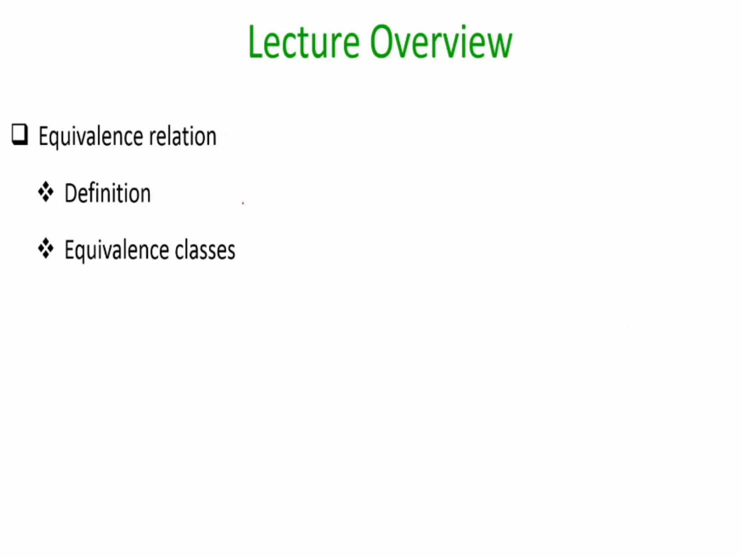
Discrete Mathematics
Prof. Ashish Choudhury
Department of Mathematics and Statistics
International Institute of Information Technology, Bangalore

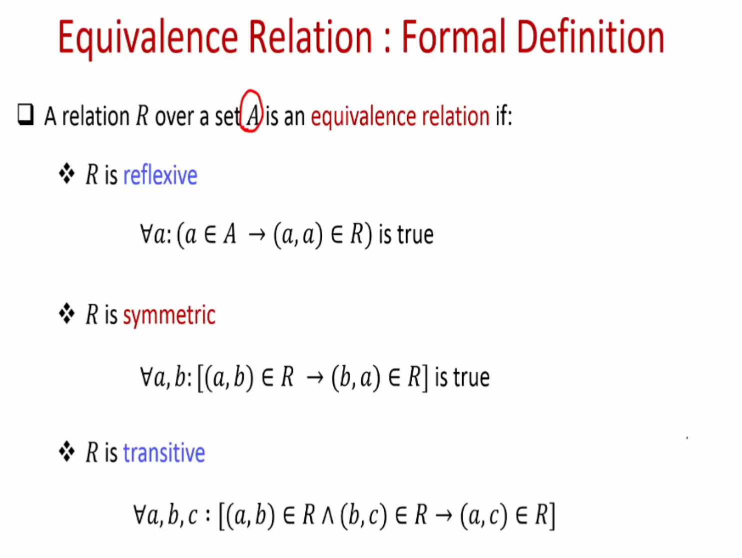
Lecture -21
Equivalence Relation

(Refer Slide Time: 00:24)



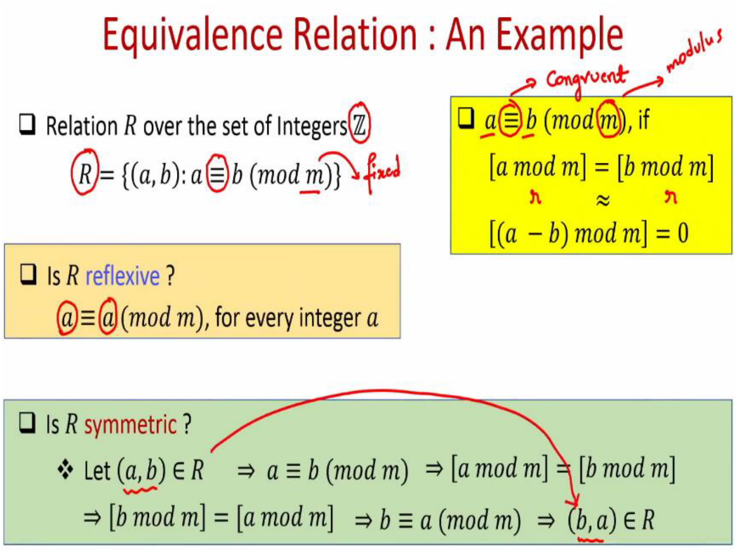
Hello everyone, welcome to this lecture on equivalence Relations. And just to recap in the last lecture we discussed some special types of relations like Reflexive Relations, Symmetric Relations, Asymmetric relations, Anti Symmetric Relations, Transitive Relations. So, in this lecture we will introduce a special type of relation called as equivalence Relation and we will see the definition of equivalence classes.

(Refer Slide Time: 00:44)



So, what is the formal definition of an equivalence relation? It is a relation R over a set A which satisfies three properties namely the relation should be reflexive, the relation should be symmetric and the relation should be transitive. It should satisfy all these three properties. If any of these three properties is not satisfied, the relation will not be called as an equivalence Relation.

(Refer Slide Time: 01:11)



So, let us see an example. So, I define a relation over ℤ here and by relation here is that an integer a will be related to integer b if $a \equiv b \ mod \ m$.

We say an integer a and integer b are congruent, they are congruent with respect to modulo m if the remainder which I obtained by dividing a by the modulus m is exactly the same as the remainder which I obtained by dividing b by the modulus m. So, m is the modulus here. I mean the divisor and you are dividing a by m and b also by m, and if you get the same remainder, then we say that a and b are kind of equivalent in the sense they have the property that they give you the same remainder when divided by this modulus m.

$$R = \{(a, b): a \equiv b \; mod \; m\},$$

Where m is a fixed modulus.

If $a \equiv b \; mod \; m, \rightarrow (a - b)$ is completely divisible by m, it gives you 0 remainder.

If $a \equiv r \; mod \; m$ and $b \equiv r \; mod \; m, \rightarrow (b - a) \equiv (r - r) mod \; m \equiv 0 \; mod \; m.$

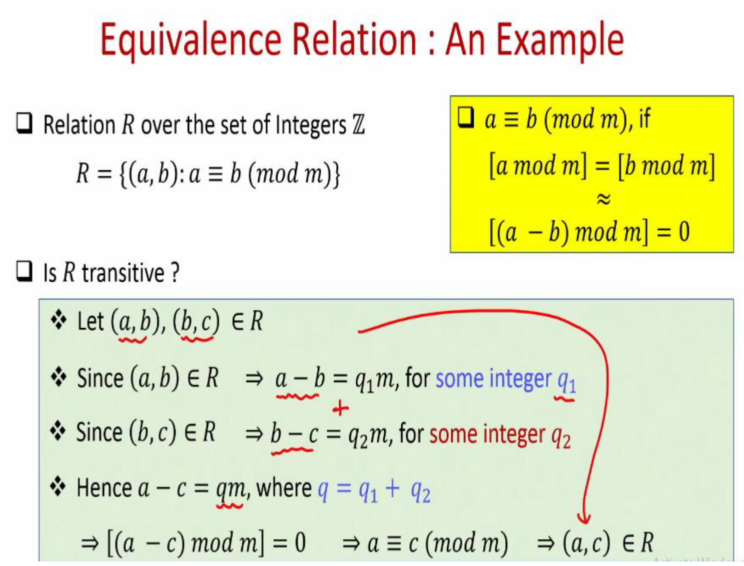
Now my claim is that this relation R is an equivalence relation. It satisfies the property of reflexive relation, Symmetric relation and transitive relation.

So, let us prove that so is the relation R reflexive? Answer is yes. Because, $(a - a) \equiv 0 \; mod \; m.$ You divide a whatever remainder you obtain by dividing a by m the same remainder you obtained by dividing a again by m. So, in that sense a is always congruent to a modulo m.

The relation R is also symmetric, we can prove that. For proving the symmetric property, I assume that consider an arbitrary pair of integers (a, b) where $a \equiv b \; mod \; m \rightarrow b \equiv a \; mod \; m.$

So, what I have proved here is (b, a) is present in the relation R. That means the integer b is related to the integer a as per my relation R. So, what I have proved is starting with the premise that (a, b) is present in the relation R. I can conclude that (b, a) is also there in relation R. That proves my relation R is symmetric.

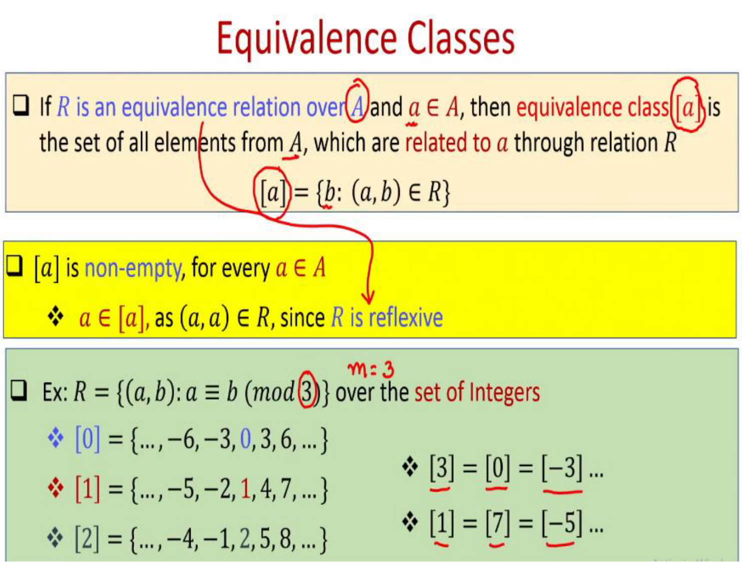
(Refer Slide Time: 05:09)



Now let us prove that the relation R is transitive as well. So, for proving the transitivity property I have to show that, if I have a related to b in my relation and b related to c in the relation, then I have to show that the integer a is related to integer c. And I have to show this for any arbitrarily chosen a, b, c. So, since a is related to integer b, that means a is congruent to b or equivalently (a – b) is completely divisible by the modulus m.

So, I can say that $a - b = q_1 \cdot m$. In the same way, since the integer b is related to integer c, that means integer b is congruent to integer c or equivalently b - c is completely divisible by m. Or in other words $b - c = q_2 \cdot m$. Now what I can say here is if I add these two equations here, I get that $a - c = (q_1 + q_2) \cdot m$. That means a - c is completely divisible by the integer m, which in other words means that $a \equiv c\ mod\ m$. That means the integer a is related to integers c. And that proves that your relation R is transitive as well.

(Refer Slide Time: 06:47)



So, that this is an example of an equivalence relation. So, now let us define equivalence classes. Imagine R is an equivalence relation over some set A. And now consider an element $a \in A$. Then the equivalence class of A which is denoted by this notation you have the square bracket and within that you have the element a. So, the equivalence class of $[a] = \{b: (a, b) \in R\}$, consist of all the elements from the set A which are related to this element a as per the relation R.

Formally, this equivalence class is a set it will be a subset of your set A. It will be having all the elements $b \in A$ such that a is related to b. That is equivalence class of an element a. And now this equivalence class satisfies some very nice properties. The first trivial thing to check here is verify here is that you take the equivalence class of any element, it will be non- empty.

There will be at least one element which is always guaranteed to be present in the equivalence class of any element a. And that element is the element a itself, $a \in [a]$. Because the element a is always related as per the relation R because the relation R is an equivalence relation and since it is an equivalence relation it is reflexive. If it is a reflexive element, every element is related to itself.

So, the element a will always be present in its equivalence class and hence a equivalence class A will never be an empty set. Let me demonstrate what exactly equivalence class looks like with an example. So, I consider this relation R over set of integers ℤ where an integer is related to integer b if $a \equiv b \ mod \ 3$. So, m = 3 here, we already proved in the previous slide that this relation is an equivalence relation.

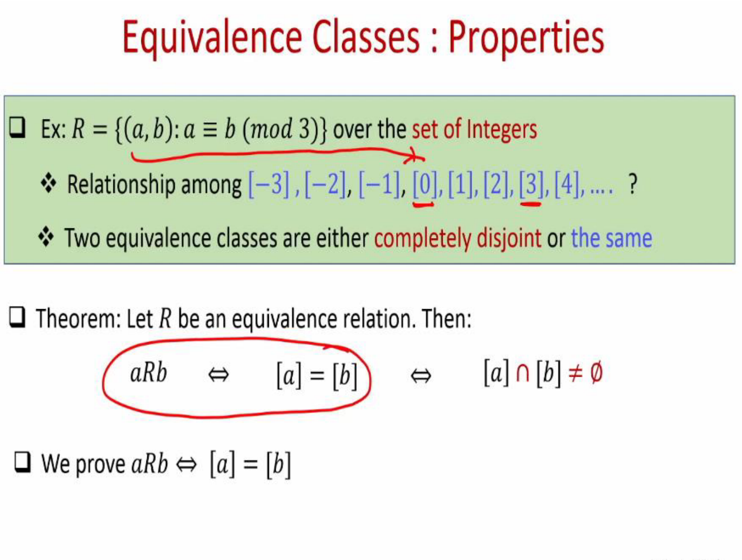
So, what will be the equivalence class of 0? So, $[0]$, so my a = 0 here, so equivalence class 0 will have all the elements b, empty all the integers b such that 0 is related to those integers b. And it easy to see that equivalence class of $[0]$ will be 0, will have definitely 0. Because 0 is related to 0 because 0 is congruent to 0 modulo 3. And equivalence class of 0 will have 3, 6, 9 and these integers because 0 is related to 3 and you have 0 related to 6 and so on.

In the same way you have 0 related to -3 you have 0 related to -6 and so on. Because 0 is congruent to -3, 0 is congruent to -6 modulo 3 and so on. So, $[0] = \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}$. all the integer multiples of 3. What about $[1]$? Definitely the element 1 will be present in the its equivalence class.

And apart from that we will have the integers 4, 7, 10 and so on. And on the negative side we have the elements -2, -5, -8 and so on present in the equivalence class of 1. Because all these integers are related to the integer 1 as per the relation R. $[1] = \{\dots, -8, -5, -2, 0, 4, 7, 10, \dots\}$. In the same way the $[2] = \{\dots, -7, -4, -1, 0, 2, 5, 8, \dots\}$.

Now if you see closely here, it turns out that $[3] = [0] = [-3]$ and so on will be the same. That means the equivalence class of all the integer multiples of 3 will be same. In the same way the $[1] = [7] = [-5]$, any equivalence class of any integer of the form 3 k + 1 are same and so on.

(Refer Slide Time: 11:23)



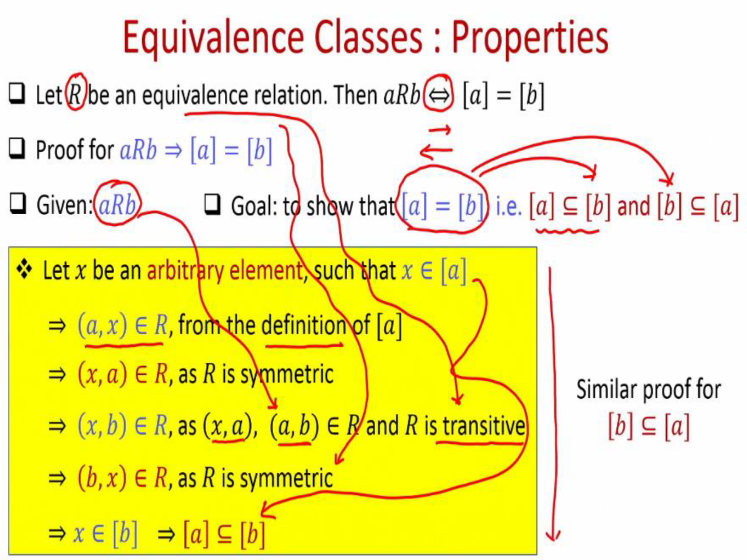
So, what we are observing here is that even though we have equivalence class of every integer possible here. So, these are the various equivalence classes. And this is an infinite list. It turns out that if we closely look here we find that the two equivalence classes in this sequence are either same or they are completely disjoint. So, for instance, if I consider [0] and [1], there will be no common element, there will not be any integer which is present simultaneously in [0] and [1] as per the relation R. You cannot have an integer b such that $b \equiv 0\ mod\ 3$ as well as simultaneously $b \equiv 1\ mod\ 3$.

Whereas if you consider [0] here and [3], they will be exactly same. They will have exactly the same elements. So, it turns out that this property that 2 equivalence classes are either completely disjoint or they are completely same is not present, this property does not hold only with respect to this equivalence relation, this special equivalence relation it holds in general for any arbitrary equivalence relation which is a very interesting property.

So, more formally we can prove that if you are given any equivalence relation, any arbitrary equivalence relation over an arbitrary set then a is related to b iff they are equivalence classes are same and the equivalence classes as $[a] = [b]$ if and only if $[a] \cap [b] \neq \phi$. Or in other words if $[a] \cap [b] = \phi$, then $[a] \neq [b]$.

And of course, we can prove we can apply the transitivity property and say that if a is related to b then $[a] \cap [b] \neq \phi$. So, there are 2 by implications involved here. I am going to prove one of the by implications and I leave the proof for the other by implication, for you it is very simple, it follows the proof of the first by implication. So, I am going to prove this by implication. I will prove that if R is an equivalence relation and if a is related to b then $[a] = [b]$.

(Refer Slide Time: 14:11)



Now since this is a by implication, I have to prove the implication in both the directions. So, I prove the first implication in the forward direction namely I assume that a is related to b. And then under this assumption I have to show that they are equivalence classes are same. So, since a is related to b. So, this is what is given to me and my goal is to show that $[a] = [b]$.

Equivalence class of a is a set, equivalence class of b is a set. So, I want to prove here that two sets are equal. So, to prove that two sets are equal I have to show that they are mutually subsets of each other. That is what is the definition of equality of two sets. So, proving that equivalence class of a is equal to equivalence class of b boils down to proving these two things. That $[a] \subseteq [b]$ and vice versa, given that a is related to b.

And how do I prove that a set is a subset of another set? I prove it by showing that you take any element x in the first set, it is present in the second set. So, I take an arbitrary element x belonging to the first set here. The first set here is $[a]$. I have to show that the same $x \in [b]$ as well. How do I do that? Since $x \in [a]$, I can say that $(a, x) \in R$. That means a is related to the element x because that is what is the definition of $[a]$.

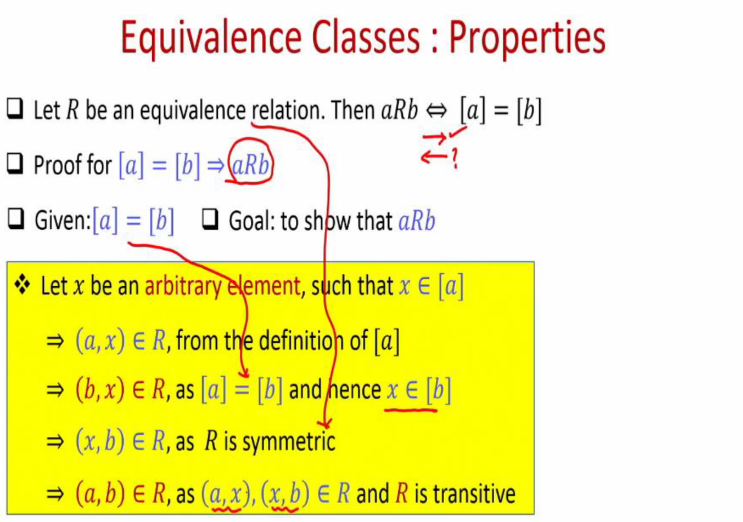
Now, I am also given that R is an equivalence relation and if R is an equivalence relation, then one of the requirements from an equivalence relation is that it should be symmetric. And if relation R is symmetric and if $(a, x) \in R$, then $(x, a) \in R$ as well.

Now, I have $(x, a) \in R$ and as per my hypothesis here $(a, b) \in R$. And since my relation R is transitive, why? Because my relation R is an equivalence relation. The transitivity property ensures that $(x, b) \in R$. Now since $(x, b) \in R$, I can again apply the fact that my relation R is symmetric because it is an equivalence relation.

So, I get that $(b, x) \in R$. And if $(b, x) \in R$, then as per the definition of an equivalence class, the element $x \in [b]$. That means starting with the premise that $x \in [a]$, I have shown that $x \in [b]$ as well. Which proves that $[a] \subseteq [b]$.

And I can apply a similar proof to show that $[b] \subseteq [a]$. So, you start with some arbitrary element $x \in [b]$ and again applying similar steps that we have done here, we have used here. You can show that the same element $x \in [a]$. And that will show that $[a] = [b]$.

(Refer Slide Time: 17:56)



So, that proves the implication in the forward direction this we have done. Now, let us prove the implication in the reverse direction. So, assuming R is an equivalence relation and assuming that the equivalence class of a and b, are same I have to show that a is related to b. So, for this I start with some arbitrary element $x \in [a]$ then as per the definition of equivalence class of a, it means that a is related to x as per the relation R.

And since it is given that $[a] = [b]$.That means the element x will be present in the equivalence class of b as well. Then as per the definition of equivalence class, it means that x is related to b as well. Since b, x is present in my relation, I can say that x, b is also present in my relation. Because R is symmetric and why R is symmetric? Because my relation R is an equivalence relation.

Now I can apply the transitivity property here on (a, x) and (x, b). So, I have (a, x) present in the relation I have (x, b) in the relation and by applying the transitivity property, I get (a, b) present in the relation. So, that proves the implication in the other direction. Remember there is another by implication which I am leaving for you to prove. And that will establish the theorem that we have stated in couple of slides back.

That brings me to the end of this lecture. Just to recap, in this lecture we introduced the notion of equivalence relation and we also introduced the notion of equivalence classes. We established important property that the equivalence classes are either disjoint or they are completely same, thank you.