Discrete Mathematics
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Module No # 05
Lecture No # 28
Examples of Countably Infinite Sets

Hello everyone welcome to this lecture on examples of countably infinite sets.

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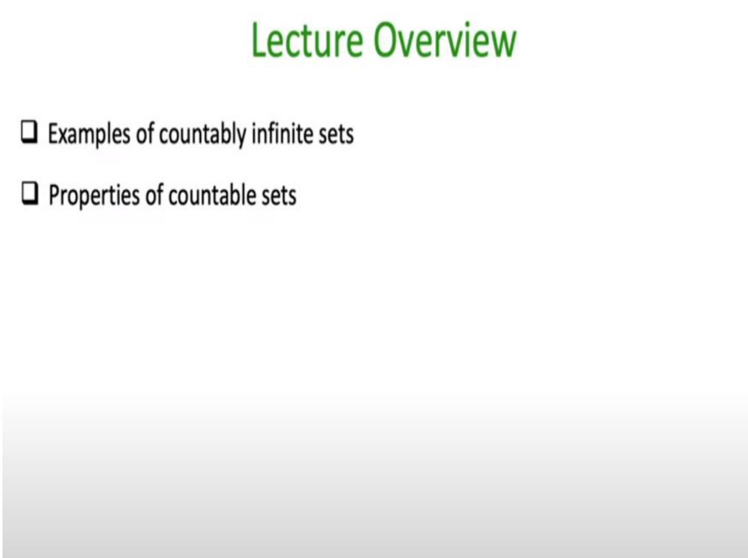

So just to recap in the last lecture we introduced the notion of countable and uncountable sets. Countable sets are those sets whose cardinality is either finite or whose cardinality is same as the set of positive integers. So the plan for this lecture is as follows. We will see several examples of countably infinite sets and we will also discuss some properties of countable sets specifically in the context of infinite sets.

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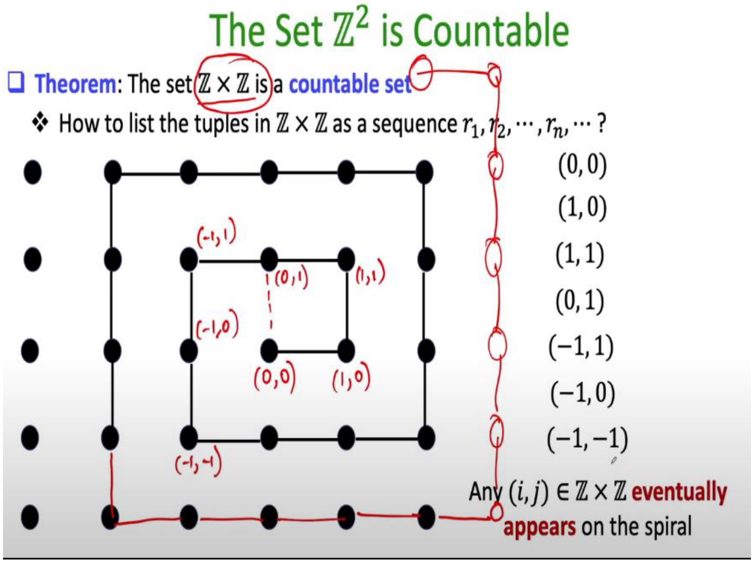

So we first prove that the Cartesian product of the set of integers is a countable set. So again this might look non-intuitive, you have many elements in the Cartesian product of the set of integers compared to the set of integers itself because when I say that Cartesian product it is going to consist of all ordered pairs of the form (i, j) where i can be any integer, j can be any integer. But what this theorem says is that the number of elements in the set ℤ x ℤ is same as the number of elements in the set of positive integers.

So we are going to prove that. So remember in the last lecture we proved that whenever you want to prove that an infinite set is countable either you gave an explicit bijection between that set and the set of positive integers. Or you give a well-defined sequence or a rule according to which you specify or list down the elements of the given set which you want to prove to be countably infinite. And argue that every element in that set will appear in the sequence that you are specifying.

So what we will do is to prove this theorem we are going to show a sequence or a way to enumerate all the elements of the set ℤx ℤ. But the question is how exactly we find out one such sequence? So that we do not miss any element of set ℤ x ℤ. So the idea is very clever here what we do is, So since we are considering the Set ℤ x ℤ it is nothing but the collection of all points in your 2 dimensional plane.

So imagine that you have that infinite 2 dimensional plane where you have all the points belonging to the $\mathbb{Z}$ x $\mathbb{Z}$. And our goal is basically to give an enumeration of all the points in that infinite plane such that the enumeration should be well defined and we do not miss any point in the enumeration process. So here is the enumeration process I start with the center namely coordinate (0, 0) which is the element of $\mathbb{Z}$ x $\mathbb{Z}$. So imagine this is your (0 ,0) this point.

Then my next point is which I am going to enumerate in my sequence; which I am going to list down in my enumeration is the point (1, 0). That means I move from my current point 1 unit to the right hand side then from my current point I move 1 unit in the positive direction and get the point (1, 1) and list it in down. And then I traverse or go 1 unit to the left hand side from the current point so I will get the point (0, 1).

And now I cannot come down because if I come down then I will be coming to the element (0, 0) which I have already listed down which I do not want to do. So what I am going to do is instead of going down from (0, 1) I will continue left further 1 unit. And due to that I will get the point (-1, 1) and I will list it down. And now I will come down because if I come down from my current point the point which I am going to get I have not enumerated it already.

So I will get a new point will be (-1, 0) and then I continue this process I go down further 1 unit and obtain the point (-1, -1). And then I will make this whole trip again. So what was the trip? I started with (0, 0) go right go up go left left down down and then I will again make this circular rotation. So what I will do is from my current point I will go right 1 unit again right 1 unit again right 1 unit. And then go up up. and then continue this process.

So I will be next enumerating this… and the next point and then I will go up….. and then continue left. So this is the process which I will follow and the idea here is that if I enumerate the various points in this infinite 2 dimensional plane according to the procedure that I have demonstrated here, any point in this infinite 2 dimensional plane will eventually appear along the spiral. That's the idea here, you will not miss any point in the infinite 2 dimensional plane.

So that is why this is a valid enumeration of all the elements in the Cartesian product of the set of integers, which shows now; that the set of the Cartesian product of the integers of all points in infinite 2 dimensional plane is a countable set.

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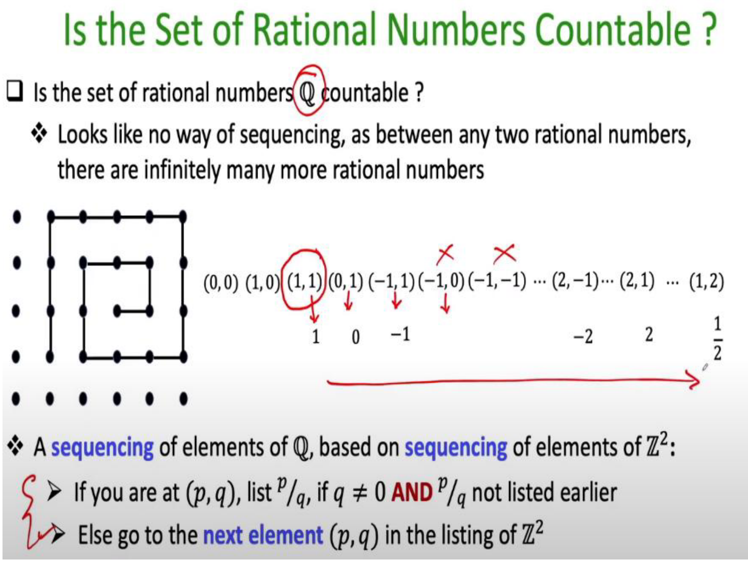

Now, we will see next whether the set of rational numbers which I denote by this ℚ notation is countable or not. Now intuitively it might look the answer is no because definitely rational numbers is a super set of the set of the integers. And looks like there is no way of sequencing because the fundamental fact about rational numbers is that you take any 2 rational numbers there are infinitely many more rational numbers between the same 2 rational numbers.

That means if I consider 2 rational number x and y between x and y there are infinitely many rational numbers. So how exactly we are going to list down or sequence all possible rational numbers. So looks like that is not possible. But what we can do is we can show a very clever enumeration of the set of rational numbers which will prove that the set of rational numbers is a countable set.

And the sequencing that we are going to see here will be based on the sequencing of the elements of the point in the 2 dimensional integer plane based on enumerating all the points along the spiral that we had been seen in the last slide. So just to recall, this was the enumeration of the set of all elements or points in the set ℤ x ℤ. And based on this enumeration we will get an enumeration of the set of all rational numbers.

So the idea is if we consider any rational number and if it is a rational number it will be of the form p / q, where p is some integer and q is some integer and q will not be 0. So the idea is you traverse or you follow the enumeration of all the elements in the set $\mathbb{Z}$ x $\mathbb{Z}$ namely this enumeration here. And based on this enumeration you come up with an enumeration of the elements in the set of rational numbers as follows.

If you are at a point (p, q), then you list down the rational number (p / q) in your enumeration provided q is not 0 because if q is 0 definitely that is not a rational number. And the rational number (p / q) is not listed earlier as per your enumeration. Else you go to the next element (p, q) in the listing of $\mathbb{Z}^2$ that is the idea. So what I am saying is demonstrated as follow. So if I apply the rule on (0, 0) so if I start with (0, 0) so my p is 0 and q is 0.

So my rule says that if q is 0 do not do anything go to the next element. And my next (p, q) is (1, 0) and again q is 0. So my rule says do not do anything. Then I go to my (p, q) which is (1, 1) and I will be applying the first rule here because here q is not 0 namely q is 1 and my (p / q) is (1 / 1) which is the rational number 1 and which is not yet listed. So that is why I will list down the element 1 then I will go to the next (p, q). q is not 0.

So again will be applying the first rule and (p / q) will be 0 in this case. Then my (p / q) will be (-1 / 1) which has not been listed earlier. So I will list it down. Then my next (p / q) is not defined because q is 0 so ignore this. Then if I go next my (p / q) is (-1 / -1) which is nothing but 1 and which has been already listed. So that is why I will apply the rule in the else part. So that is why I will miss this element as well and if I continue then when I go to the element (2, -1) it will be (2 / -1) which is the rational number -2 which has been not listed earlier.

So now you can see even though there are infinitely many rational numbers if I follow these 2 rules of enumerating the rational numbers I will not be missing any rational number because you take any rational number it will be of the form (p / q). And if will be eventually listed down in the sequencing that I have specified here. So that means we now have a method of listing down all the elements of the set $\mathbb{Q}$ in a well-defined fashion and that is why this set of rational numbers will be a countable set.

It has infinitely many element but we can count it in the sense we can sequence down we can write down all the elements in that set. So this will be the sequencing of the elements in the set of rational numbers.

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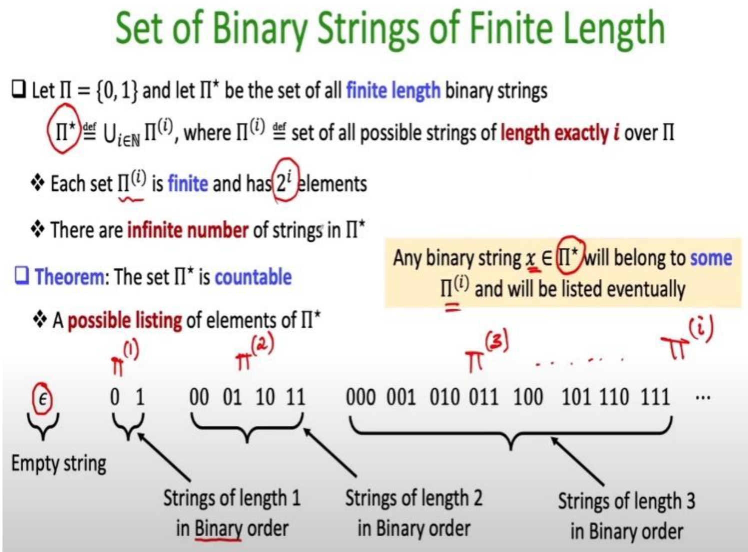

Now let us consider the set of binary strings of finite length. What exactly that means? So, imagine a set Π consisting of 2 elements namely the element 0 and 1 and why 0 and 1? Because I am considering binary strings so, binary strings will be just a string of 0's and 1's. And I used this notation Π* to denote the set of all binary strings of finite length. What do I mean by that? So more formally Π* is defined to be the union of the sets $\Pi^{(i)}$ where i is within parenthesis.

And i belongs to the set of natural numbers namely i ranges from 0 to infinity. And what is this set $\Pi^{(i)}$ within parenthesis it is set of all possible binary strings. So I should specify here it is the set of all possible strings of length exactly i over the alphabet Π. And since Π consists of only symbol 0 and 1 what does $\Pi^{(i)}$ denote? It denotes the set of all possible binary strings of length exactly i.

So if I consider the set $\Pi^{(1)}$ it will have only the binary strings of length 1. So it will have only 2 elements. If I consider the set $\Pi^{(2)}$ it will have all binary strings of length 2 and so on. So what is this set Π*? It is the set which is obtained by taking the union of $\Pi^{(1)}$ $\Pi^{(2)}$ and so on including $\Pi^{(0)}$ and where $\Pi^{(0)}$ denotes all possible binary strings of length 0.

So we use this special notation ε to denote the set of to denote an empty binary string. So based on this fact it should be now clear that each subset $\Pi^{(i)}$ is finite. Why it is finite? Because it has exactly 2^i elements because $\Pi^{(i)}$ denote a set of all possible binary strings of length exactly i and I can have 2^i such binary strings.

And if I take the union of all such sets I get the set Π*. So it is easy to see that the set Π* is an infinite set because the number of element is infinite. But it is the union of several subsets where each subset is finite in the sense it has finite number of elements. So now the question is, is this the Π* countable even though it has infinitely many elements it has infinite number of binary strings can we numerate down all such strings in a well defined fashion.

So the answer is yes we can prove that the set Π* is indeed countably infinite. And what we will do is to prove this theorem we will show a possible valid listing of the elements of Π*. And the idea is to arrange or list down all the elements of Π* according to their length. So we start with the length 0 strings and length 0 string will be the empty string denoted by the special notation ε.

Then we will go and enumerate or list down all valid string, binary strings of length 1. And there are multiple strings of a particular length. We arrange them according to the binary order. So for example here we have 2 possible binary string of length 1 : 0 and 1. But since numerical is 0 is less than 1 we will list down 0 followed by 1. Then so basically what I am saying here is that you go to the set $\Pi^{(1)}$ and list down the elements of the set $\Pi^{(1)}$ in binary order.

Next go to the set $\Pi^{(2)}$ and it will have 4 elements, list down those elements in binary order. So we have 0 listed first followed by 1, followed by 2, followed by 3 and continue this process. Next go the set $\Pi^{(3)}$ which will have 8 strings list down those strings in binary order and so on. So, why this is a valid listing? The idea is you take any binary string x belonging to Π*.

It will have finite length because as per the definition of Π*, x will be belonging to some set $\Pi^{(i)}$. We do not know what exactly is that index i it depends upon the number of bits or number of characters in your string x but it is a well defined value that means x belongs to some $\Pi^{(i)}$. And eventually after listing down all the elements in the set $\Pi^{(0)}$ to $\Pi^{(i-1)}$ when we will be listing down the elements of the set $\Pi^{(i)}$ x will appear somewhere in our listing.

So we will not be missing the element x. And we know that after some step eventually the chance for x will come as per this listing to be listed down in our enumeration. So that is why this is the valid enumeration it shows that the set Π* even though it has infinite number of elements it is possible to list down those elements in a well-defined way and hence proving that the set Π* is countable.

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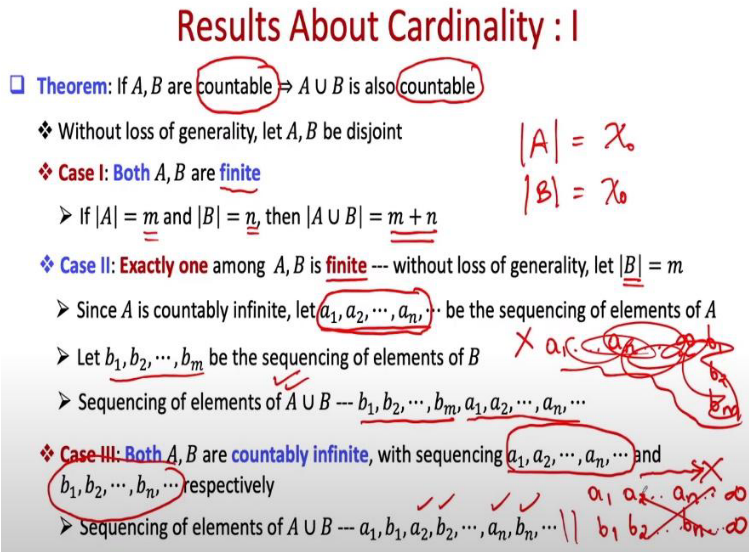

So till now we had been seen several infinite sets and magically we have proved that they have same cardinality. Now we will prove some general results about the cardinality of sets both with respect to finite sets and infinite sets. So the first theorem is that if you have 2 sets A and B and if they are countable then their union is also countable. So I am not saying anything about the number of elements in the union A and B.

Of course, what I am saying is that it is always possible to list down the elements of A U B. So how we are going to prove it? First of all there might be a possibility that A and B are not disjoint but to keep our proof simple without loss of generality we assume that A and B are disjoint. The proof can be simply adapted for the case when A and B are not disjoint. Now we can have various cases depending upon whether A, and B are countably finite or countably infinite.

So the theorem statement was that A and B are countable and the definition of countable set is that either its cardinality is finite or its cardinality is same as $\aleph_0$. So we can have 3 possible cases here. Case 1 when both A and B are finite that means say if the cardinality of A is m and the cardinality of B is n then it is easy to see that the cardinality of union of A and B will be m + n which is a finite number and hence A U B is also countable.

Case 2 is when exactly 1 of the set A and B is finite whereas the other set is countably infinite. Now again we can have 2 possible cases depending upon which of the 2 sets is countably infinite. So what we can do is we assume without loss of generality that it is A set which is countably infinite that means the cardinality of A is $\aleph_0$. And the set B is finite that means it has exactly m number of elements where m is some natural number.

So what we are now going to show is that even in this case the union of A and B is countable. Of course the union of A and B will have infinite number of elements because A is infinite here. But what we are going to do is we are going to show here a valid sequencing for the elements in the, set A U B. So the idea here is that since A is countably infinite, it will have some valid sequencing of its elements. So let that valid sequencing be a_1, a_2, a_n and so on. And of course we know that set B has m number of elements.

So let the elements be, b_1 to b_m. So what we do is we list down the elements of; we can say that we can list down the elements of A U B as follows. First list down the elements of B set which are finite in number, m in number followed by the elements of the set A. Now you might be wondering why we cannot do the following. Why we cannot we enumerate the elements of the set A first and then followed by the elements of the set B. My claim is that this is not a valid sequencing of the elements in the set A U B.

Why it is not valid is because since you are first listing down the elements of the set A you do not know when you are going to return and come back and list down the elements of the set B because that sequencing of the elements of the set A is an infinite process and you can get stuck there forever. So you do not know when exactly you will finish the process and will come and start listing down the elements of the set B.

So now what I mean here is that if I ask you, can you tell me where exactly b_1 is going to appear in this sequence? You cannot tell me because we do not know when exactly we will finish listing down the elements of the set A and then we will come to and list down element B. But if I consider this sequencing which I have listed here I know where exactly when exactly the element will appear irrespective of whether it belongs to the set A or the set B in the sequencing.

If you are asking me to specify where exactly an element from the set B belongs to I can give you that position. Whereas if you ask me where exactly is the position of an element from the A set in this sequencing again I can tell you that it will appear somewhere because as per my assumption that element has some position in the sequencing of the elements of the setting. So that is why it is this sequencing which is valid and not this sequence.

The third case is when both A and B set are infinite and countable, because I am assuming my A and B sets are countable and if A; and B are both infinite that means both the cardinality of A as well as the cardinality of B is $\aleph_0$. And I want to show that A U B is also countable by giving you a valid sequencing for the elements in the union of A and B. So since A and B are countably infinite they will have individual valid sequencing of the elements of the respective sets.

So, image that this is the sequencing of the elements of the set A and this is the sequencing of the elements in the set B. We want to find out a valid sequencing of the elements in A U B so that we do not miss any element in the union of A and B. And we know when exactly an element in the union of A and B is going to appear in the sequence. So a valid sequencing of the elements in union of A and B is as follows.

First list down first element in the A sequence followed by the first element in the B sequence. Then go and list down the second elements of A sequence and B sequence and like that continue and write down or list down the nth element in the A sequence and B sequence and so on. So now you can see that you ask me any element belonging to the union of A and B it will eventually appear in this sequence. It would not be the case that we get stuck infinitely for listing down the element.

Whereas if I would have listed down first elements of the A set and then list down the element of B set then this is not a valid sequencing for the elements of the A U B why? Because if you now ask me, when exactly I am go to list or when exactly I am going to see b_1 and the sequencing. I do not know because the process of listing down all the elements of A set is a never ending process. So we do not know when exactly we will finish that process and come and write down or list or find the element b_1.

So that is why this is not valid sequencing but the same problem would not happen with the sequencing that I have specified here namely listing down the elements of A and B sets alternatively because it does not matter what is the element in the A U B that will appear somewhere in the a sequence or in the b sequence depending upon whether it belongs to the, A set or the B set. Accordingly since I am listing down the elements of A set and B set alternately it will appear somewhere in this sequence.

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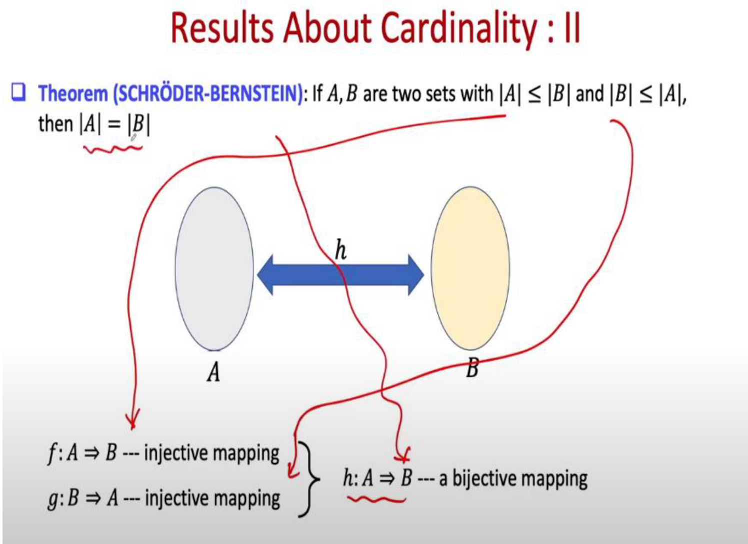

Now second interesting result about the cardinality theory is what we called as the Schroder Bernstein theorem which says the following. If you have 2 sets such that the cardinality of A is less than equal to cardinality of B and simultaneously the cardinality of B is less than equal to the cardinality of A. Then we can conclude that both set A and B have the same cardinality. In terms of function what we are saying here is that if |A| is less than equal to |B| then as per the definition we have an injective mapping say the mapping f from the set A to B.

And since the cardinality of B set is less than equal to the cardinality of A set we also have a injective function say g from the set B to set A. Now if we have these 2 individual injective mappings, what this theorem basically tells you is that, using the injective mappings f and g you can come up with the bijective mapping between the set A and B. That is the idea behind the proof of this theorem.

However the proof is slightly involved and due to the interest of time I will not be going through the proof of this theorem. But this is a very important theorem which we should keep in our mind. What this theorem basically says is, if you want to show that the cardinalities of 2 sets are same then one way of doing that is you show one injective mapping from the first set to the second set and another injective mapping from the second set to the first set.

That automatically will conclude that you have; you can have a bijection also between the 2 sets. And if you have a bijection between the 2 sets then as per the definition of (cardinality) equality of 2 sets they have the same cardinality not equal not sorry equality of the cardinality of the 2 sets, they have the same cardinality. The sets A and B might different. They may have different elements. But cardinality wise they will be the same.

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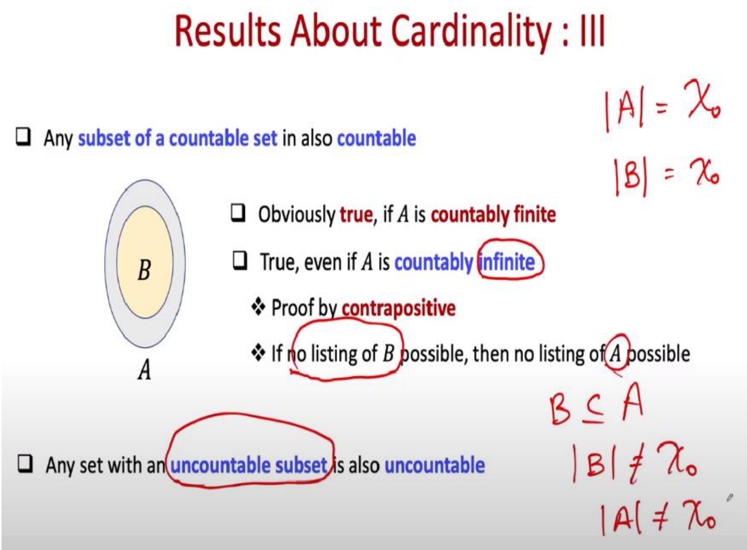

Now the third result about the cardinality is the following. If I take any subset of a countable set then it should be also countable. So, there are 2 cases the above statement is obviously true if the set A is a countably finite set. That means if the set A has say n number of elements and if I take subset B of the set A of course the cardinality of B will be upper bounded by n. So this statement is obviously true statement is obvious also true even if the set A is infinite but countable.

So I can prove that even if the set A is infinite but countable that means its cardinality is $\aleph_0$. Then the cardinality of any subset B of that set A is also $\aleph_0$, we can prove that. The idea behind the proof is as follows. We can prove the theorem by contrapositive and the simple way to understand the proof is that if the set B is not countable. That means if it is not countable that means it is not possible at all to list down the elements of the set B.

So if you do not know any method of listing down the elements of the subset B how come it is possible to list down the elements of the superset A. And that goes again the assumption that my set A is countably infinite. If I assume that my set A is countably infinite that means I know how to list down the elements of set A in a well defined fashion. So that is the proof for this fact. So as a consequence of this statement I can also state that if you have any set which has an uncountable subset, then the set is also uncountable.

So what I am saying is that if you have a scenario where B is the subset of A and you do not know how to list down the elements of the set B that means the cardinality of B is not $\aleph_0$. Then I can conclude that the cardinality of A is also not $\aleph_0$. This is because if I do not know how to list down the elements of set B I do not know how to list out the elements of the set superset A as well.

Because while listing down the elements of the superset A I need to list down the elements of the subset B as well. But I do not know how to list down the elements of the subset.

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