Discrete Mathematics
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Lecture -14
Tutorial 2: Part II

Hello everyone. Welcome to the second part of tutorial 2.

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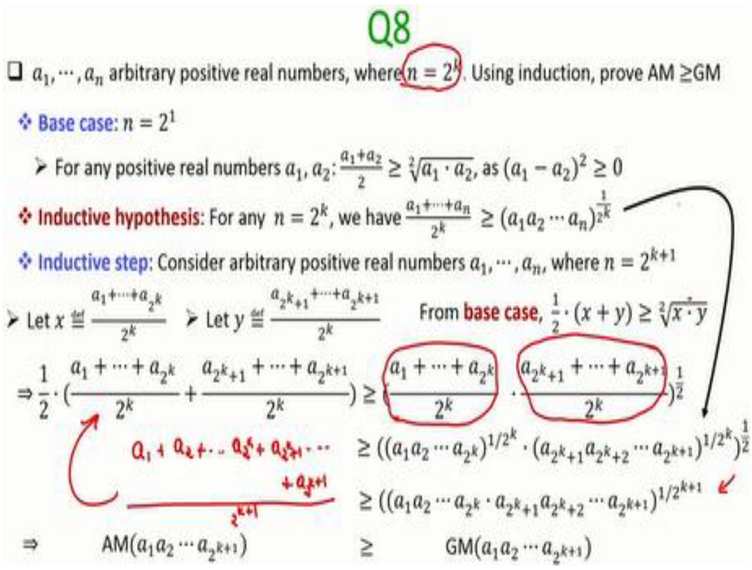

So we start with question number 8. So here you have to use proof by induction to show that if you are given n arbitrary positive real numbers, where n is some power of two. Then their arithmetic mean is greater than equal to geometric mean and this is true for any collection of n arbitrary positive real numbers provided n is some 2^k. So this is a universally quantified statement because I am making this statement for all n where n is equal to 2^k.

So I have to prove a base case and I take the base case where k equal to 1, k equal to 1 my statement true for two positive real numbers. This is two if you remember the proof mechanisms we use a backward proof mechanism to prove that arithmetic mean of any two positive real numbers is greater than equal to their geometric mean. So the base case is true. Now assume the statement is true for any collection of n numbers n positive real numbers where n is 2^k.

And, since it is true for n equal to 2^k that means this expression or this inequality holds. The left hand side is your arithmetic mean. The right hand side is your geometric mean. The geometric mean will be the $(2^{k)}$th root of the product of a_1 to a_n which can be rewritten and the form that is given here. Now, I want to prove the statement to be true for next higher power of n, next higher power of n is $2^{(k+1)}$.

So to do that what I do here is the following let me define x. So you are given now a collection of $2^{(k + 1)}$ numbers which you can split into two parts. You can consider the first collection of 2^k numbers and the next 2^k, next 2^k numbers in the list. So this part has 2^k elements, this part also has 2^k elements this is your list a_1 to a_n. So what I do here is I define the quantity x and y here.

So x is the Arithmetic mean of the first 2^k elements in my collection and my y here is the arithmetic mean of the next 2^k elements and the collection, you can verify here. Now what I know is that I can treat x and y as two numbers they will be positive real numbers and I know that from my base case the arithmetic mean of x and y will be greater than equal to the geometric mean of x and y.

So if I expand this, the arithmetic mean of x and y will be as follows; so this is your x, this is your y. The arithmetic mean will be x + y over two, so one over two I am taking outside and geometric mean will be $(xy)^{1/2}$ namely the square root of x times y. So this is your x and this is your y, this is what I get from the base case. Now, what I do is I apply the inductive hypothesis on my right hand side.

Since I am assuming my inductive hypothesis to be true, I know that arithmetic mean of any 2^k elements is greater than equal to its geometric mean. So the portion that I have circled here it is an arithmetic mean of 2^k numbers, so that is greater than equal to the geometric mean of those 2^k numbers. In the same way the y here can be considered as arithmetic mean of 2^k numbers.

And that will be greater than equal to the geometric mean of those 2^k numbers the one over two outside remains as it is. Now what I can do is I can take this 2^k $1/2^k$, $1/2^k$ appearing in the exponent all together outside and multiply the first 2^k numbers and the next 2^k numbers, this is just plain simplification. But, now if I rearrange everything or reinterpret everything the left hand side is nothing but the arithmetic mean of 2^{k+1} numbers.

Because the arithmetic mean of 2^{k+1} numbers here will be $a_1 + a_2 \ldots + a_{2^k} + a_{2^k + 1}$ upto $a_{2^{(k+1)}}$ whole over $2^{(k+1)}$ which I can rewrite in this form and your right hand side expression is nothing but the geometric mean of the same $2^{(k+1)}$ elements. So now you can see that I am using the base case here as well as the inductive step here to prove the inductive hypothesis here to prove my inductive step. So that completes your question number 8.

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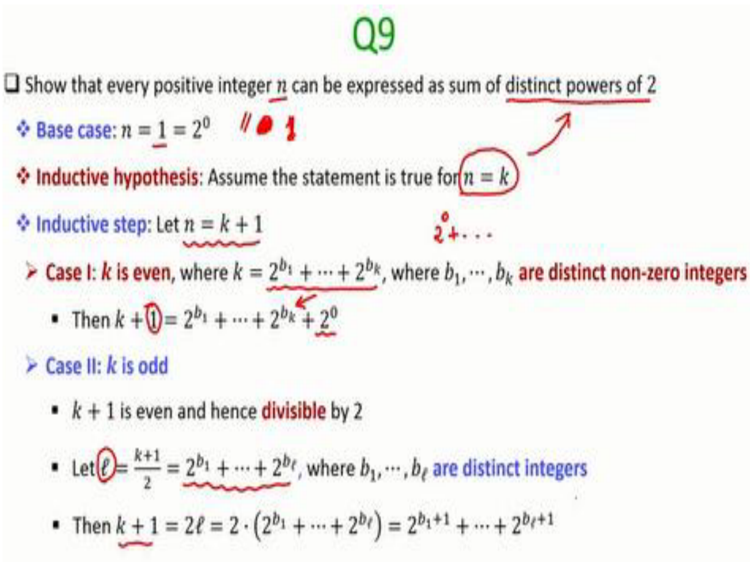

In question 9, you asked to prove that every positive integer n can be expressed as sum of distinct powers of two basically this is a fundamental fact that we learn that you take any positive integer, it has a binary representation and a binary representation of that number is nothing but sum of distinct powers of two. So that is what we want to prove here, you want to prove it can be always possible to represent any positive integer is.

And the powers of two here will be distinct which is equivalent to saying that every positive integer has a unique or distinct binary representation. So we will prove it by induction because this is a universally quantified statement my base case will be n equal to 1. If my integer n is 1, then I can represent this as 2^0. Namely the binary representation is zero, the binary representation zero corresponds to 2^0 here.

If n is equal to one the binary representation is one here and a binary representation one corresponds to 2^0 here. Let us take the inductive hypothesis here assume the statement is true for n is equal to k that means you give me any integer k where k is arbitrary, it can be expressed as the sum of distinct powers of 2 or it has a distinct binary representation. Making this hypothesis, assuming this hypothesis to be true I will prove the inductive step and will show a unique binary representation for the integer k + 1.

So, how do I proceed here? So I use proof by cases. Case one, if k is even; now if k is even and since k has a binary representation a unique binary representation namely k is expressible as sum of distinct powers of two and let that sum of distinct powers of two will be this. My claim here is that 2^0 is not present in this existing binary representation or sum of distinct powers of two in the representation of k because k is even.

If k is even you cannot have 2^0 also present along with other powers of two when you express k as the sum of distinct powers of two because that will imply your k is odd. So since 2^0 is missing in the representation of k and I want to represent k + 1 what I can do is I can take the representation of k and to that I add a new power of two namely 2^0 that will give me the representation of k + 1.

And since all the existing powers of two in the representation of k are distinct and none of those powers were zero by adding this 2^0, I am not violating my condition, which I want to prove here. My case two is when k is odd. Now, if k is odd then it follows that k + 1 will be even and if k + 1 is even that means it is divisible by 2 and k + 1 suppose it is *l* and *l* will be a number which is definitely less than equal to k.

Since it is less than equal to k that means from the inductive hypothesis, it follows that there is a binary unique binary representation for *l* and namely *l* can be represented as sum of distinct powers of two and let that sum of distinct powers of two is this. Now what I know is k + 1 is nothing but two times *l* and two times *l* can be obtained by just incrementing all the powers of two that we had in the representation of *l*.

Since the different powers of two in the representation of l were distinct each of them incremented by one will still give me distinct powers of two and now if I sum this new powers of two that will give me the integer k + 1. So now you can see here that when I am proving it for the case when k is odd, I am using a strong induction because I do not know what I cannot say it definitely l is equal to k, l is k + 1 over 2. So it is any value in the range 1 to k.

So I have to use the inductive hypothesis; I have to assume it is true for all integers in the range 1 to l and that is why it is proof by strong induction.

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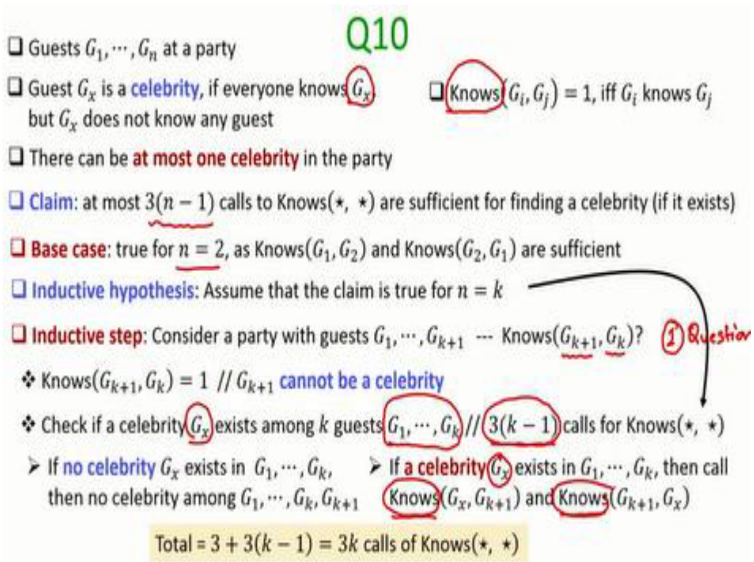

Now come to question number 10. In question number 10, you are given the following, you are having a party and n guests in a party and in the party of n people we call a guest to be a celebrity, this is my definition of a celebrity, if every guest in the party knows the guest G_x while G_x does not know any of the other guest. If that is the case then I will call the guest G_x as a celebrity and our goal here is to find out whether there exist a celebrity in the party or not.

And for doing that given here a primitive namely you are asked to ask questions of the form Knows(G_i, G_j). So if you ask, guest number i whether he knows guest number G_j then you can get the 0, 1 answer depending upon whether guest number i know is guest number j or not that means this is the only operation allowed to you. You can ask guest number i well, you know guest number j or not and you can tell you whether he knows the guest or not.

And vice versa you can ask guest number G_j, you can ask guest number j whether he knows guest number i or not and depending upon whether he knows or not you get answer 1 or 0. So the question first part of the question ask you how many celebrities can be there in a party. Well, the first thing is it is not necessary that there are exist definitely a celebrity in the party. It might be possible that all the n people know each other. In that case none of them is a celebrity.

Because in that case there exist no celebrity, because everyone knows each other because the condition of the celebrity is that the celebrity should not know any of the other guests. But if everyone knows everyone then how can the celebrity be possible? So it turns out that if at all a celebrity is there, there can be only one celebrity you cannot have two celebrities you cannot have a celebrity G_x as well as a celebrity G_y simultaneously.

Because if G_x is a celebrity then he should know G_y and if G_y is a celebrity then he should not know G_x, but since G_x is a celebrity, G_y knows G_x, because which gives you a contradiction. So you cannot have two simultaneous celebrities possible in a party. If at all there is a celebrity you can have exactly one celebrity.

So now in this question, we want to prove that in order to find celebrity in a party, it is sufficient to make at most three times n - 1 number of calls to this Knows primitive. That means you can ask at most, it is sufficient to ask at most three times n - 1 questions, asking various guests whether they know other guest or not to find out whether a celebrity exists in the party or not. So we will prove it by induction, before proceeding when I say three times n - 1 definitely for n greater than equal to two.

Because it does not make any sense this expression three times n - 1 becomes 0; if n equal to 1. So my claim here is that it is sufficient to make three times n - 1 number of calls in any party consisting of two or more people to find out whether a celebrity exist or not. So we start with the base case imagine you have only two guests G_1 and G_2. So to find out whether there exists a celebrity or not, you just have to ask two questions, whether G_1 knows G_2 and whether G_2 knows G_1 or not.

And two is definitely right for n equal to two the expression three times n - 1 is three times 2 - 1 is 3 and so you are able to find out the celebrity within the allowed limit here. So assume the statement is true for n equal to k that is my inductive hypothesis that means assume you have an arbitrary party consisting of n arbitrary guest where n is equal to k and three times k - 1 questions or calls for Knows primitive or sufficient to find out the celebrity.

Now in that party if a new guest comes where the new guest is denoted by G_{k+1}, I have to prove that I can still find out whether the celebrity exist or not, by making three times k number of calls. So this is assumed to be true, I have to show this. So here is my algorithm to find out the celebrity among this k + 1 guests. I first ask the new guest who has joined the party whether he knows the guest G_k or not and there could be two possibilities.

If indeed the new guest G_{k+1} knows the guest number G_k. Then I can rule out the possibility of guest number k + 1 to be a celebrity, because he knows someone and as a celebrity he is not supposed to know anyone. So G_{k+1} cannot be a celebrity and I have already asked one question here. Now what I do is, since G_{k+1} cannot be a celebrity if at all a celebrity is there he will be in the remaining group of k people.

So I check whether there exists a celebrity among the remaining group of k people and from my inductive hypothesis these many calls namely 3 times k - 1 calls for the Knows primitive or three times k - 1 questions are sufficient to check whether a G_x celebrity exists or not in the remaining group of k people. Now there can be two possibilities; if in the remaining group of k people no celebrity exist then I can simply say that there is no celebrity in the overall group of k + 1 people.

Whereas if I find a celebrity G_x in the group of first k people, I cannot say that he is also a celebrity even if I include the k + 1 th guest, because I have to check whether the guest G_x who is the celebrity among the first k guest knows G_{k+1} or not. So I have to now ask two more questions; mainly I have to ask the guest G_x whether he knows G_{k+1} and the same way I have to ask the G_{k+1}, whether he knows G_x or not.

And then only I can confirm whether G_x is a guest in the whole group of whole bunch of k + 1 people or not. So in this whole process how many questions I am using/asking. So, I have already asked one question here to check whether G_{k+1} was a celebrity or not. Now, I have one question here, one question here. So total three questions there and in the group of k people, I will be requiring, I will be making three times k - 1 number of calls.

So the total number of calls that I need here is summation of three and three times k - 1, which is three k that is case one.

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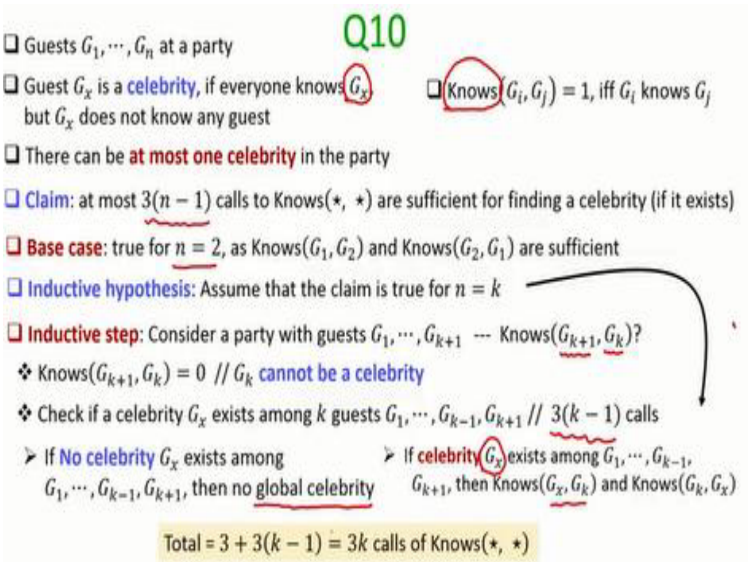

Case 2 would be the following; that guest number k + 1 does not know G_k. If the guest number k + 1 does not know guest G_k, then definitely G_k cannot be a celebrity because if at all G_k was a celebrity then he should be known by everyone but k + 1 does not know G_k. So what now I have to do is I now have to focus on the remaining k people excluding G_k. So I am excluding G_k throwing out G_k and now I am left with only k people.

And I have to check whether it there exists a celebrity in the group of these k people that I can do by making three times k - 1 number of calls. This follows from my inductive hypothesis and there could be two possible cases. If in this remaining group of k people no celebrity exist then I can say that no global celebrity exists. By global celebrity means a celebrity in the whole bunch of k + 1 people.

Whereas if I find a celebrity G_x in this reduced bunch of k people, I have to check whether that guest whether that celebrity knows G_k or not, eliminated party who cannot be the guest. Because then only I can confirm whether G_x is the global celebrity or not. So now again in this case the total number of questions that I am making that I am asking is three plus three times k - 1, which is 3 k.

So in both cases, I have shown that it is sufficient to make three times k number of calls to find out or check the possibility of a celebrity and that completes our inductive step and that proves that the claim that I made here is correct.

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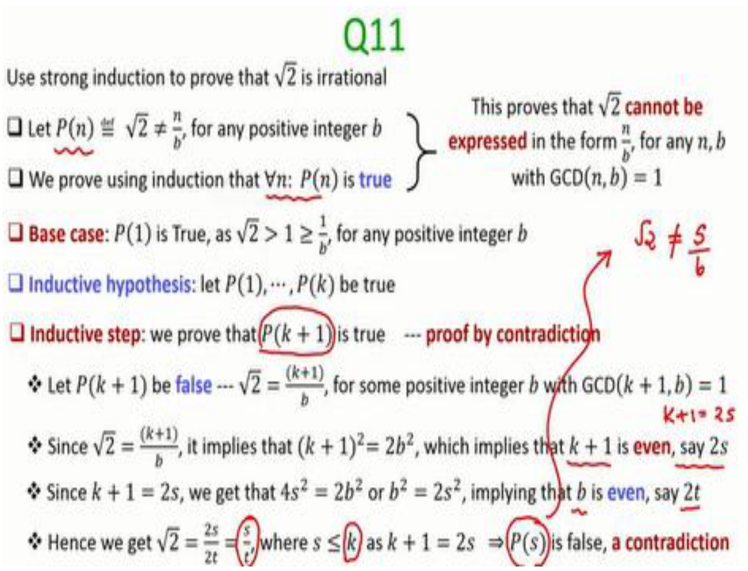

In question 11, we are supposed to use strong induction to prove that √2 is irrational. Just to recap we already proved that √2 is irrational using a proof by contradiction using proof by contradiction, but here I am asking you to do the same thing to show the same thing using strong induction so before starting the strong induction proof we have to first identify the universal statement which we are trying to make.

Remember, an induction is used to prove a universally quantified predicate. So first we have to identify what exactly is the predicate here. So the predicate P(n) here is the following; P(n) is the predicate that √2 is not equal to n/b for any positive integer b and I want to prove that this universal quantification is true using strong induction because if this universal quantification is true, that means that √2 is not equal to one over any b and it is also not equal to two over any integer b, it is also not equal to three over any integer b and in the same way it is not possible to represent √2 in the form of any n over b and if √2 is not representable in the form of any n over b that shows that as per the definition of rational numbers √2 is irrational. So how do I prove this universal quantification? I start with the base case.

I start with the base case, my base case will be when n is equal to 1 that means √2 is not equal to one over any b where b is a positive integer and this is obviously true because we know that the value of √2 is greater than one and one over any positive integer will be strictly less than or equal to one. So your √2 will be 1.44 something, something and on your right hand side is you have integers of the form one or one over two, one over three one over four, and so on.

So your left hand side is always greater than right side. So that is why your base case is true here. Now assume my inductive hypothesis is true, that means √2 cannot be represented in the form of one over b, √2 is cannot be represented in the form of 2 over b and in the same way √2 is not cannot be represented in the form of k over b.We want to prove that a statement is true even for k + 1.

Now to prove the statement is true for k + 1, I will be using a proof by contradiction and that is allowed because overall I am using an inductive proof mechanism where I have to now prove that this proposition P(k + 1) is also true, that I can prove using contradiction with the help of induction. So, since I am using proof I contradiction I will assume that the proposition P(k + 1) is false; that means √2 can be represented in the form k + 1 over sum positive integer b such that the G C D of k + 1 and b is 1.

And now I recall the proof that I used to prove that √2 is irrational using the contradiction method. So, since √2 is now assumed to be of the form k + 1 over b, I can get the conclusion that

$(k + 1)^2$ equal to 2 b^2, which means that k + 1, is even. So I can prove this and I can prove that if the square of a number is even then the number itself is even. So the same thing we did even for our earlier proof to prove √2 is irrational.

So I am not separately proving that, so since I come to the conclusion that k + 1 is even say 2 s and if k + 1 is even then I also get the conclusion that b is even namely 2 t. that means I can say that √2 can be represented in the form 2s / 2t, two cancels out and I get the conclusion that √2 is of the form s/t where s is less than equal to k, this is because I started with √2 equal to k + 1 and k + 1 is 2 s.

So, s will be definitely less than equal to k because your k + 1 is 2 s. So s is basically k + 1 over two, so definitely s is at most k and that means the proposition P(s) is false because P(s) means that √2 is not equal to s over any b. That is what is the proposition P(s), but I am getting the conclusion here that √2 is some s over positive integer that means √2 can be represented in the form s over some positive integer, that means P(s) is false here.

So, which gives me a contradiction because I assumed at P(s) is true and P(s) is true means this is true, but I get here a conclusion that √2 is equal to s/t. So these two things contradict each other.

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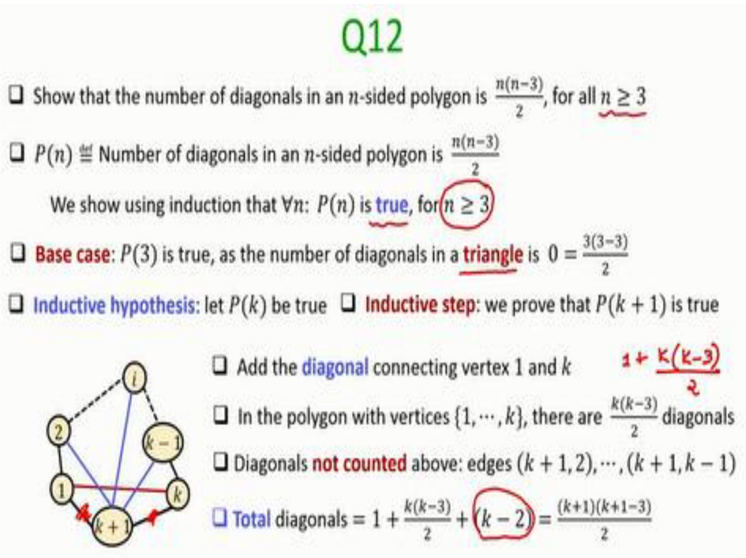

In question 12, I am supposed to find out the number of diagonals in an n sided polygon and I want to prove that it is n times n minus three over two, of course for all n greater than equal to three and I will prove it by induction. So, let us first define the predicate which we want to prove here. So the predicate here is that P(n) is true if the number of diagonals in n sided polygon is n times n minus three over two and using induction we want to prove that for all n greater than equal to three the property P is true for n.

Of course my base case will be three because I am making this statement to be true I am assuming, I am making the claim statement is true for n greater than equal to three onwards. So, of course the statement is true for any polygon with three sides because a polygon with three sides is nothing but a triangle and you do not have any diagonal in a triangle. Assume the statement is true for any polygon with k sides.

I now want to prove that the statement is true even for the polygon with k + 1 sides. So here is a polygon with k + 1 sides and I want to count the number of diagonals here. So if I add the side, if I add the vertex number one and with vertex number k, these are non adjacent vertices that constitutes one of the diagonals in this polygon of k + 1 sides. Now, if I focus on this polygon; this is now a polygon with k vertices or k sides.

And it has these many diagonals namely k times k minus three over two diagonals and these diagonals also will constitute a diagonal of the overall polygon with k + 1 sides. So, I already found these many diagonals in the overall polygon but now the question is that is are not the only diagonals. I still have diagonals which I have not included in my list and these diagonals are the diagonal obtained by connecting vertex number two with k + 1, the vertex three with vertex k + 1, vertex i with vertex k + 1 and like that vertex k - 1 with vertex k + 1.

None of this diagonals in this blue color where denoted by this blue color are included currently in my enumeration process. Now how many such diagonals are there, which I have not included yet namely the blue color was they are k - 2. Namely, I cannot count k + 1, 1, this side cannot be considered as a diagonal this is not a diagonal and this is not a diagonal. So the remaining possibilities are k - 2 number of diagonals, which are denoted by blue color.

And this gives me the total number of diagonals in the overall polygon and it comes out to be what you want to show for your inductive step, that brings me to the end of tutorial number two. Thank you.